Lesson 16 Improper Integrals.

Proper Integrals

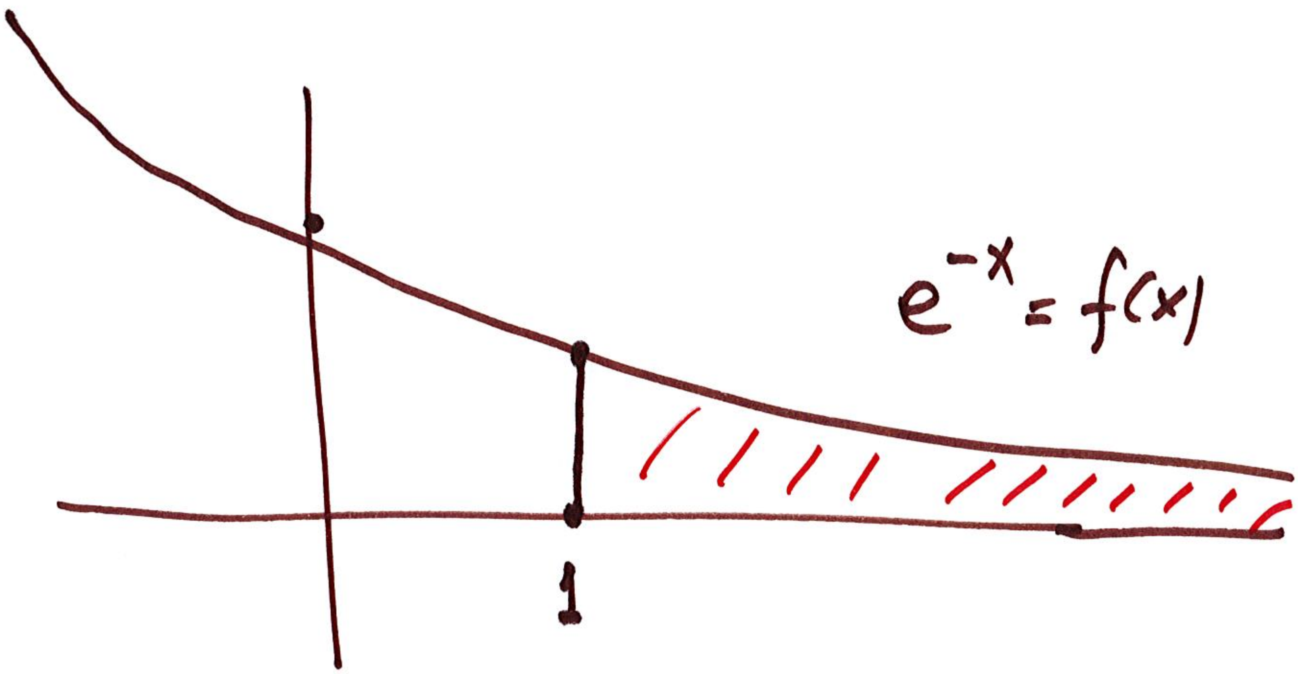
$f(x)$ continuous on $\boxed{[a,b]}$ finite interval

$\int_a^b f(x)\,dx$ is a PROPER Integral.

Ex: $\int_1^2 x^2\,dx$,

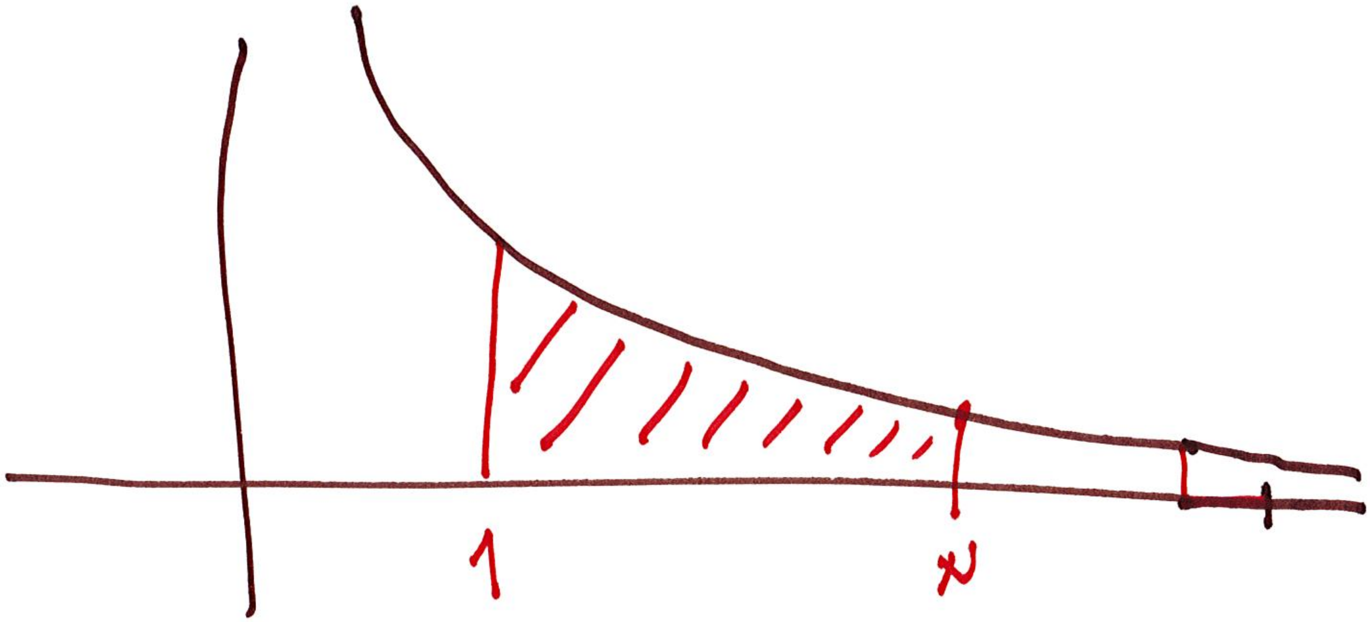
Improper Integral

$$\int_1^\infty e^{-x}\,dx$$



Definition:

$$\int_1^\infty e^{-x}\,dx = \lim_{N\to\infty} \int_1^N e^{-x}\,dx$$



$$\int_1^N e^{-x}\,dx = -e^{-x}\Big|_1^N$$

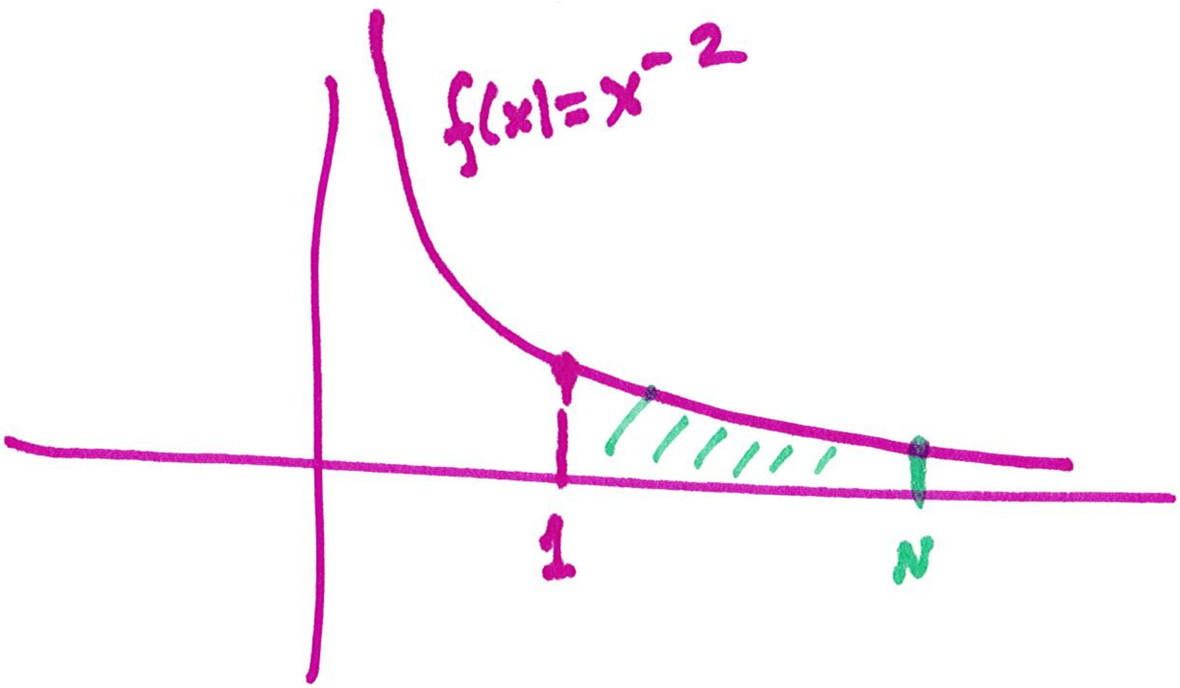
$$= -\left(e^{-N} - e^{-1}\right)$$

$$= e^{-1} - e^{-N}.$$

$$\lim_{N\to\infty} \int_1^N e^{-x}\,dx = e^{-1}.$$

$$\boxed{\int_1^{\infty} e^{-x}\,dx = e^{-1}}$$

Example: $$\int_1^\infty x^{-2}\,dx$$



$$\int_1^\infty x^{-2}\,dx = \lim_{N\to\infty}\int_1^N x^{-2}\,dx$$

$$\int_1^N x^{-2}\,dx = -x^{-1}\Big|_1^N = -\left(\frac{1}{N}-1\right)$$

$$= 1-1/N.$$

$$\lim_{N\to\infty} \int_1^N x^{-2}\,dx = \lim_{N\to\infty}\left(1-\tfrac{1}{N}\right)$$

$$= 1.$$

$$\int_1^\infty x^{-1}\,dx = \lim_{N\to\infty}\int_1^N x^{-1}\,dx$$

$$\lim_{N\to\infty}\left(\ln N - 0\right) = \infty$$

In this case we say that

$\int_1^\infty x^{-1}\,dx$ does not

converge.

<u>Question</u>: For what values

of p

$$\int_1^\infty x^{-p}\,dx$$

converge?

$$\int_1^\infty x^{-p}\,dx = \lim_{N\to\infty} \int_1^N x^{-p}\,dx$$

If $p \neq 1$

$$\int x^{-p}\,dx = \frac{x^{1-p}}{1-p} + C.$$

$p=1$ $$\int x^{-1}\,dx = \ln(x) + C.$$

$$\int_1^N x^{-p}\,dx = \frac{1}{1-p} x^{1-p} \Big|_1^N$$

$$= \frac{1}{1-p}\left(N^{1-p} - 1\right)$$

$$\lim_{N\to\infty} \int_1^N x^{-p}\,dx = \lim_{N\to\infty} \frac{1}{1-p}\left(N^{1-p} - 1\right)$$

$$\lim_{N\to\infty} N^{1-p}.$$

<u>If $p < 1$</u>, $1-p > 0$ $\quad N^{1-p} \longrightarrow \infty$ as $N \to \infty$

If $p > 1$, $1-p < 0$

$$\lim_{N\to\infty} N^{1-p} = \infty \quad \text{if } p < 1$$

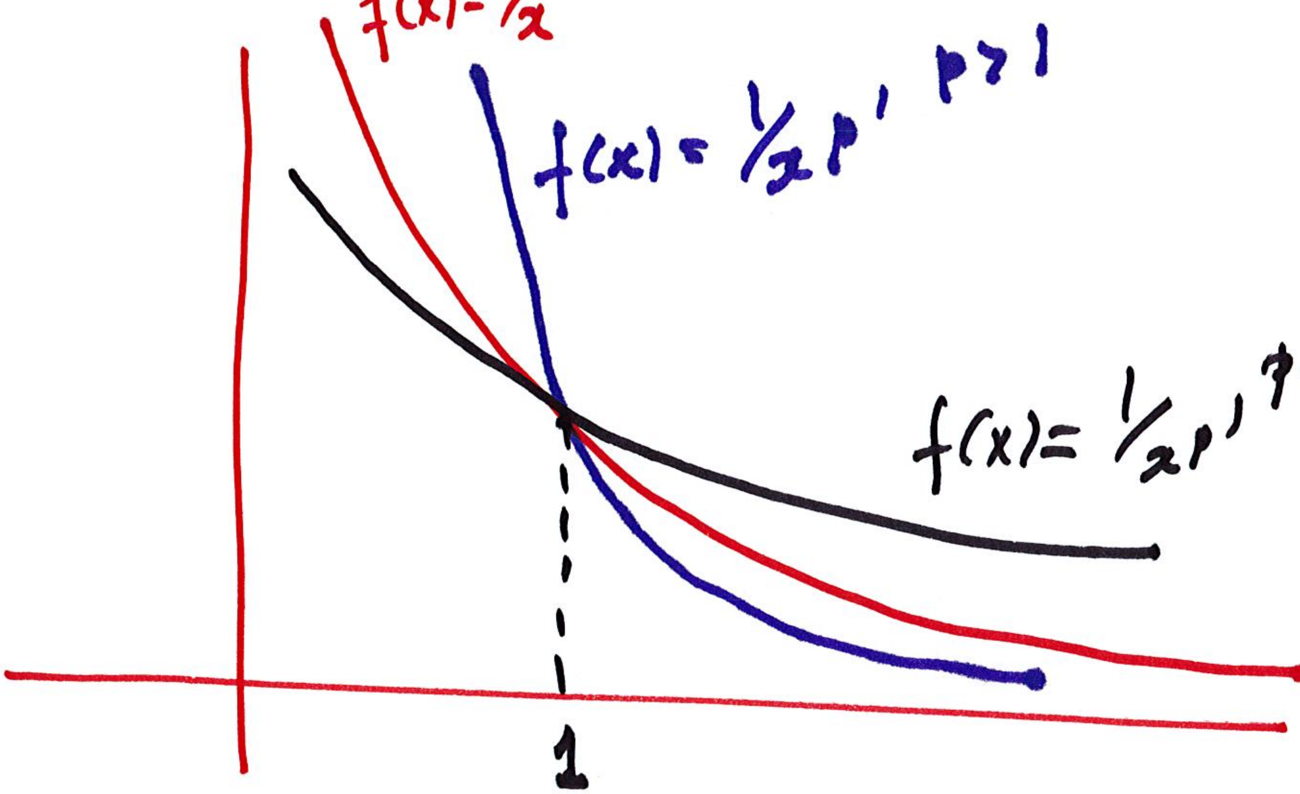
$$\lim_{N\to\infty} N^{1-p} = 0 \quad \text{if } p > 1.$$

$$\int_1^\infty x^{-p}\,dx = \lim_{N\to\infty} \frac{1}{1-p}\left(N^{1-p} - 1\right)$$

$$\int_1^\infty x^{-p}\,dx = \infty \ (\text{Diverges}) \text{ if } p<1$$

$$\int_1^\infty x^{-p}\,dx = \frac{1}{p-1} \quad \text{if } p > 1.$$

$$\int_1^\infty x^{-p}\,dx \begin{cases} \text{Diverges if } p \leq 1 \\ \\ \text{Converges if } p > 1. \end{cases}$$



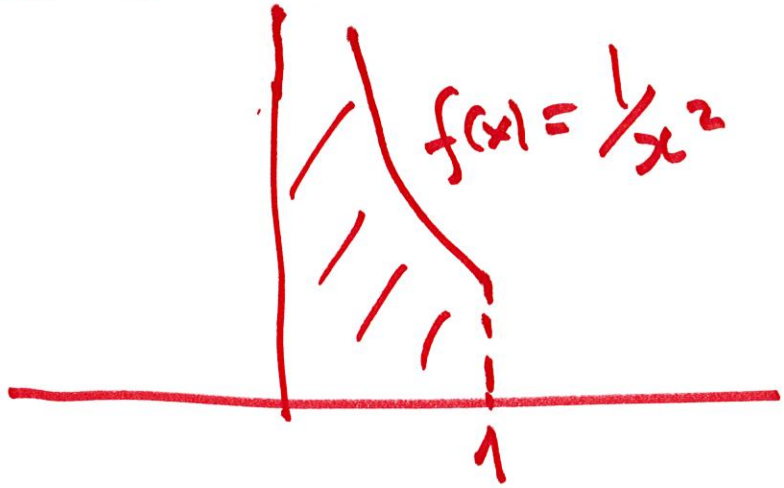
$$\int_1^2 \frac{1}{(x-1)^2}\,dx$$

When $x \to 1$ $\quad \frac{1}{(x-1)^2} \to \infty$

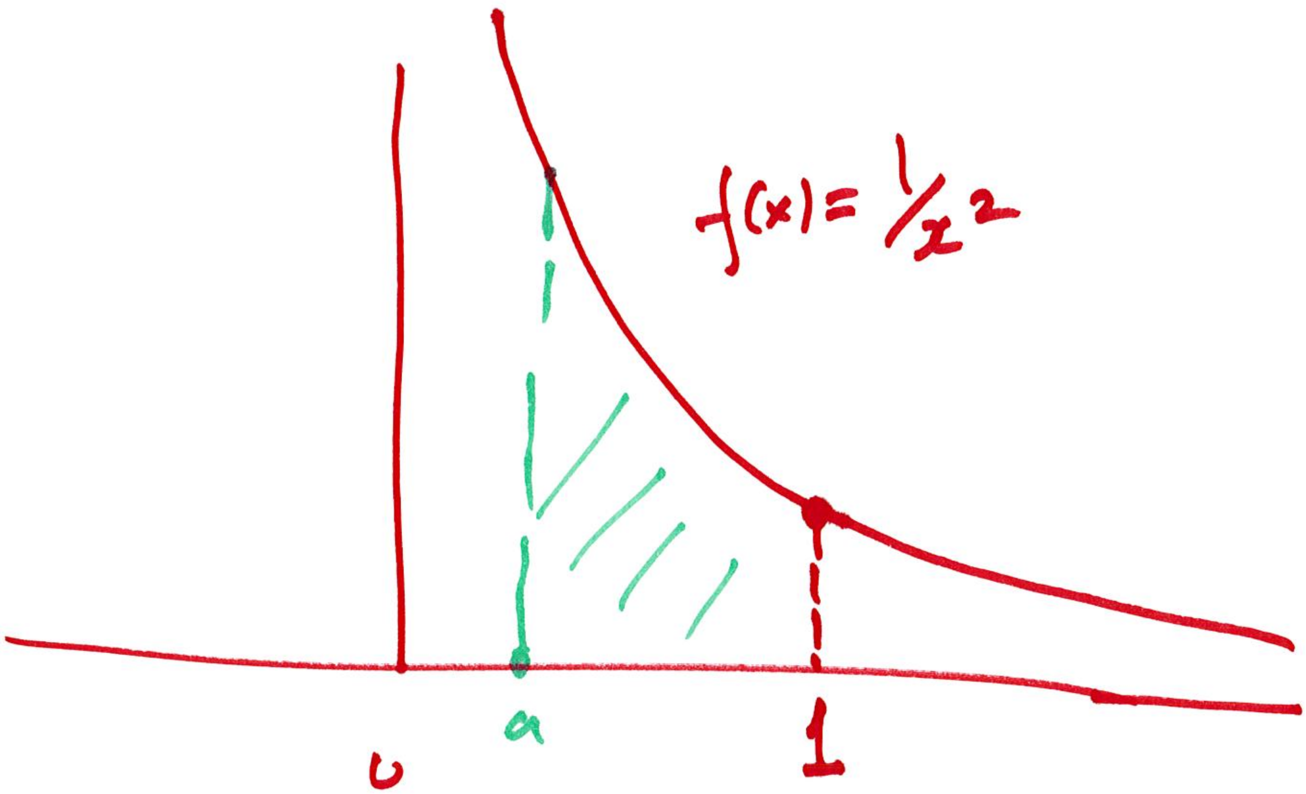
$x-1 = u$

$$\int_0^1 \frac{du}{u^2}.$$

$\frac{1}{u^2}$ is not continuous on $[0,1]$



$$\int_0^1 \frac{1}{u^2}\,du = \lim_{a\to 0} \int_a^1 \frac{1}{u^2}\,du.$$



$$\int_a^1 \frac{1}{u^2}\,du = -u^{-1}\Big|_a^1$$

$$= -\left(1 - \frac{1}{a}\right) = \frac{1}{a} - 1.$$

$$\lim_{a\to 0} \int_a^1 \frac{1}{u^2}\,du = \infty.$$

Question: For what values of p is

$$\int_0^1 x^{-p}\,dx$$

convergent?

$$\int_0^1 x^{-p}\,dx = \lim_{a\to 0} \int_a^1 x^{-p}\,dx$$

$p \neq 1$

$$\int_a^1 x^{-p}\,dx = \frac{x^{1-p}}{1-p}\Big|_a^1$$

$$= \frac{1}{1-p}\left(1 - a^{1-p}\right).$$

$$\lim_{a \to 0} a^{1-p} = \begin{cases} 0 & \text{if } p<1 \\ \infty & \text{if } p>1 \end{cases}$$

When $p=1$

$$\int_a^1 x^{-1}\,dx = \ln|x|\Big|_a^1$$

$$= \ln 1 - \ln a = -\ln a.$$

$$= \ln(1/a) \to \infty.$$

<u>Conclusion:</u>

$$\int_0^1 x^{-p}\,dx \begin{cases} \text{converges if } p<1 \\ \text{diverges if } p\geq 1. \end{cases}$$

The comparison test for Improper Integrals

Does

$$\int_1^{\infty} \frac{dx}{x^2 + 20x + 100}$$

converge or diverge?

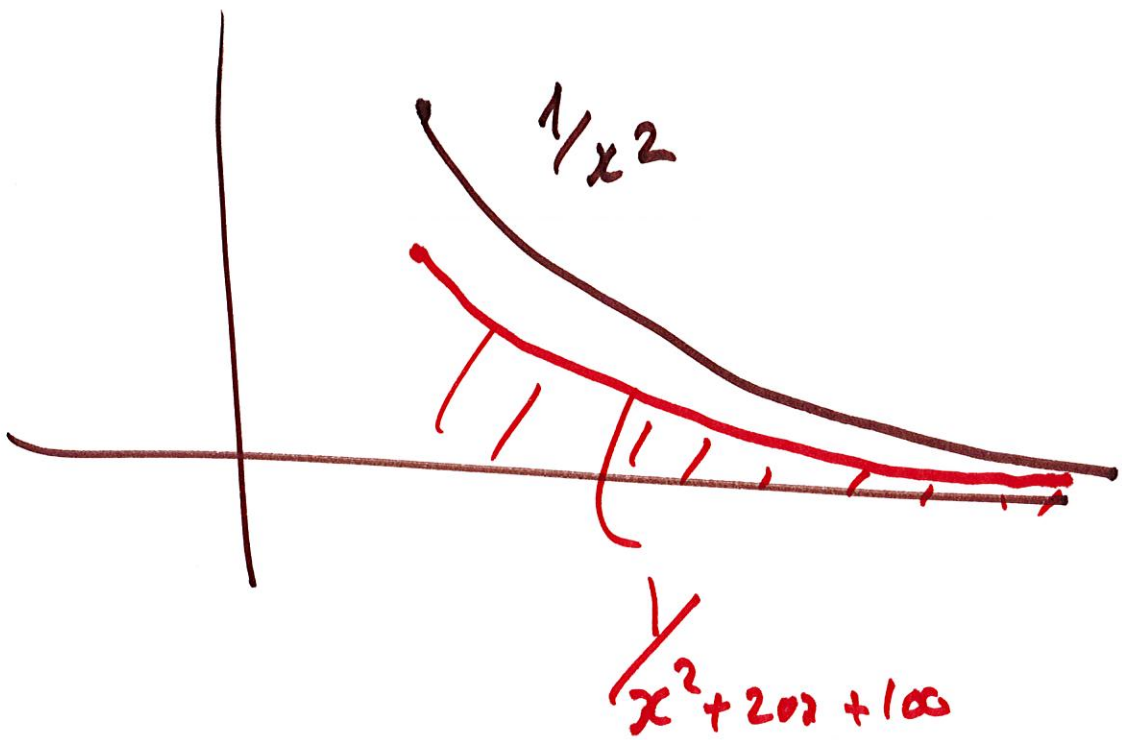
We know that

$$\int_1^{\infty} \frac{dx}{x^2}$$ converges.

for $x \geq 1$

$$\underbrace{x^2 + 20x + 100} > x^2.$$

$$\frac{1}{x^2+20x+100} < \frac{1}{x^2}.$$



$\int \frac{1}{x^2+20x+100}\, dx$ converges