Review Exam 1

Section 12.1 : 17, 18, 19, 20, 31, 46.

Section 12.3 : 15, 16, 23, 24, 29, 30, 39, 40

Section 12.4 : 4, 5, 6, 27, 28, 35, 36, 37,

Section 6.1 : 13, 14, 17, 20, 22,

Section 6.2 : 1, 3, 4, 6, 8, 10, 11, 12

Section 6.3 3, 4, 9, 10, 15, 16

Section 6.4 : 7, 8, 9, 20, 21,

Section 7.1 : 3, 4, 5, 11, 15, 51, 52,

Section 7.2 : 1, 2, 4, 5, 11, 16, 22,
25, 28, 29

Lesson 11 Section 7.3

Exam 1: Lessons 1 to 10.

Lesson 11 Trig Substitutions

Example:

$$\int \frac{dx}{x\sqrt{1-x^2}}.$$

Idea

Make a substitution that turns this into a Trig Integral.

Trig Identities

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$\sqrt{1-x^2} \underset{\text{resembles}}{\sim} \sqrt{1-\cos^2\theta}.$$

$$\int \frac{dx}{x\sqrt{1-x^2}}$$

Try $x = \cos\theta.$

$$1-x^2 = 1-\cos^2\theta$$
$$= \sin^2\theta.$$

$$\boxed{\sin\theta = \sqrt{1-x^2}.}$$

$$dx = -\sin\theta\, d\theta$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{-\sin\theta \, d\theta}{\cos\theta \cdot \sin\theta}$$

$$= - \int \frac{d\theta}{\cos\theta}$$

Remember that $\frac{1}{\cos\theta} = \sec\theta$

$$\boxed{\int \frac{dx}{x\sqrt{1-x^2}} = - \int \sec\theta \, d\theta}$$

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\underbrace{\left|\sec\theta + \tan\theta\right|}_{\text{Epress this in terms of } x.} + C$$

$$x = \cos\theta, \quad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\left|\frac{1}{x} + \frac{1}{x}\sqrt{1-x^2}\right| + C.$$

Idea is to Substitute

$x = \cos\theta$

when one has an integral

involving $\sqrt{1-x^2}$.

$$\int_0^{1/2} x^3 \sqrt{1-4x^2}\, dx.$$

If we set $x = \cos\theta$

$$1 - \underset{\sim}{4}x^2 = 1 - 4\cos^2\theta.$$

Set $\boxed{x = \tfrac{1}{2}\cos\theta.}$

$dx = -\tfrac{1}{2}\sin\theta\, d\theta.$

$$1 - 4x^2 = 1 - \cancel{4} \cdot \frac{1}{\cancel{4}} \cos^2\theta$$

$$= 1 - \cos^2\theta = \sin^2\theta$$

$$\boxed{\sqrt{1-4x^2} = \sin\theta}$$

$x = \frac{1}{2}\cos\theta$

$x = 0, \ \frac{1}{2}\cos\theta = 0$

$\cos\theta = 0$

$x = \frac{1}{2}, \ \cos\theta = 1$

$$\int_0^{1/2} \underbrace{x^3}\, \underbrace{\sqrt{1-4x^2}}\, dx =$$

$$= \int_{\pi/2}^{0} \frac{1}{8}\cos^3\theta \cdot \sin\theta \cdot -\frac{\sin\theta}{2}\, d\theta.$$

$$= \frac{1}{16}\int_0^{\pi/2} \cos^3\theta \cdot \sin^2\theta\, d\theta.$$

$$\int \cos^3\theta \cdot \sin^2\theta \, d\theta.$$

$$= \int \cos\theta \cdot \cos^2\theta \cdot \sin^2\theta \, d\theta.$$

$$= \int \cos\theta \, (1-\sin^2\theta) \cdot \sin^2\theta \, d\theta$$

$$u = \sin\theta, \quad du = \cos\theta$$

$$= \int (1-u^2)\, u^2 \, du$$

<u>Conclusion:</u>

$$\int_0^{1/2} x^3 \sqrt{1-x^2}\, dx = \frac{1}{16} \int_0^1 (1-u^2)\, u^2 \, du$$

$$= \frac{1}{16}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$\boxed{\begin{array}{l}\text{Integrals Involving} \\ \sqrt{b^2 - a^2x^2} \\ \text{Set } \quad x = \frac{b}{a}\cos\theta .\end{array}}$$

$$\cos^2\theta + \sin^2\theta = 1.$$

Divide by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$$\int \frac{dx}{\sqrt{1+x^2}}$$

$x = \tan\theta$, $\quad 1+x^2 = 1+\tan^2\theta = \sec^2\theta$

$$\boxed{\sqrt{1+x^2} = \sec\theta.}$$

$dx = \sec^2\theta \, d\theta.$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta \, d\theta}{\sec\theta}$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left| x + \sqrt{1+x^2} \right| + C.$$

$$\int_0^2 \frac{dx}{(x^2+4)^2}. \qquad x = 2\tan\theta$$

$x=0,\ \tan\theta = 0$

$x=2,\ 2 = 2\tan\theta$

$\tan\theta = 1$

$$x^2+4 = 4\tan^2\theta + 4$$

$$= 4(\tan^2\theta + 1)$$

$$= 4\sec^2\theta$$

$$\boxed{x^2+4 = 4\sec^2\theta}$$

$dx = 2\sec^2\theta\, dx$

$$\int_0^2 \frac{dx}{(x^2+4)^2} = \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta\, d\theta}.$$

$$= \frac{1}{8}\int_0^{\pi/4} \frac{1}{\sec^2\theta}\, d\theta.$$

$$\frac{1}{\sec\theta} = \cos\theta$$

$$\frac{1}{\sec^2\theta} = \cos^2\theta.$$

$$= \frac{1}{8}\int_0^{\pi/4} \cos^2\theta \, d\theta.$$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \frac{1}{8}\int_0^{\pi/4} \frac{1+\cos 2\theta}{2} \, d\theta.$$

$$= \frac{1}{16}\int_0^{\pi/4} d\theta + \frac{1}{16}\int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= \frac{\pi}{64} + \frac{1}{16} \cdot \frac{1}{2} \sin 2\theta \Big|_0^{\pi/4}$$

$$= \pi/64 + \frac{1}{32}$$