End of Lesson 12

Example

$$\int_{\sqrt{2}}^{2} \frac{dt}{t^3\sqrt{t^2-1}}.$$

Recall that

$\sqrt{1-t^2} \longrightarrow$ Substitution $t = \cos\theta$ or $t = \sin u$

$\sqrt{1+t^2} \longrightarrow t = \tan\theta$

When we have $\sqrt{t^2-1}$

$t = \sec\theta.$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1.$$

$t^2 - 1$, set $t = \sec\theta$

$$t^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$$

$$\sqrt{t^2-1} = \tan\theta.$$

$$\int_{\sqrt{2}}^{2} \frac{dt}{t^3\sqrt{t^2-1}} = \int_{\pi/4}^{\pi/3} \frac{\sec\theta \cdot \cancel{\tan\theta}}{\sec^3\theta \cdot \cancel{\tan\theta}}\, d\theta.$$

$t = \sec\theta$, $dt = \sec\theta \cdot \tan\theta$

$\sec\theta = \frac{1}{\cos\theta}$, $\sec\theta = \sqrt{2}$, $\frac{1}{\cos\theta} = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = \pi/4$

$t = 2$, $\cos\theta = 1/2$, $\theta = \pi/3$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\, d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\, d\theta.$$

$$= \int_{\pi/4}^{\pi/3} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right) \Big|_{\pi/4}^{\pi/3}.$$

$$\int \frac{dx}{\sqrt{x^2 - 10x - 24}}.$$

Reduces this to an integral $\displaystyle\int \frac{dt}{\sqrt{t^2 + a^2}}$

or $\boxed{\displaystyle\int \frac{dt}{\sqrt{t^2 - a^2}}}$ or

$$\int \frac{dt}{\sqrt{a^2 - t^2}}.$$

$$x^2 - 10x - 24 = (x-5)^2 - 25 - 24$$
$$= (x-5)^2 - 49.$$

$$\int \frac{dx}{\sqrt{x^2-10x-24}} = \int \frac{dx}{\sqrt{\underbrace{(x-5)}_{t}{}^2-49}}$$

Substitution: $\boxed{x-5 = 7\sec\omega}$

$$(x-5)^2-49 = 49\sec^2\omega - 49$$

$$= 49(\sec^2\omega - 1)$$

$$= 49\tan^2\omega$$

$$\boxed{\sqrt{(x-5)^2-49} = \sqrt{49\tan^2\omega} = 7\tan\omega.}$$

$$x-5 = 7\sec\omega \quad , \quad dx = 7\sec\omega\tan\omega.$$

$$\int \frac{dx}{\sqrt{(x-5)^2-49}} = \int \frac{7\sec\theta \cdot \tan\theta \, d\theta}{7\tan\theta}$$

$$= \int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C.$$

$$= \ln\left| \frac{x-5}{7} + \frac{1}{7}\sqrt{(x-5)^2-49} \right| + C.$$

Integration By Partial Fractions.

Idea:

$$\int \frac{dx}{(x-2)(x+1)}$$

$$\int \frac{dx}{x+1} = \ln|x+1| + C$$

$$\int \frac{dx}{x-2} = \ln|x-2| + C.$$

Question: $$\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \text{ ?}$$

$$\frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

$$= \frac{1}{(x-2)(x+1)} \quad \text{for every } x$$

$$A(x+1) + B(x-2) = 1.$$

$$Ax + A + Bx - 2B = 1$$

$$\boxed{x(\underbrace{A+B}) + A - 2B = 1.}$$

$$A + B = 0$$

$$A - 2B = 1.$$

$$3B = -1 \quad , \quad B = -1/3$$

$$A = 1/3.$$

$$\frac{1}{(x+1)(x-2)} = \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$$

$$\int \frac{dx}{(x+1)(x-2)} = \int \left[\frac{1}{3(x-2)} - \frac{1}{3(x+1)} \right] dx$$

$$= \frac{1}{3} \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{x+1}$$

...

$$= \frac{1}{3} \ln |x-2| - \frac{1}{3} \ln |x+1| + c.$$

$$2x+1 = Ax^2 - Ax - 6A$$

$$+ Bx^2 - 4Bx + 3B$$

$$Cx^2 + Cx - 2C$$

$$= 2x+1 = x^2(A+B+C)$$

$$+ x(C - A - 4B)$$

$$+ 3B - 6A - 2C.$$

$$\begin{cases} A + B + C = 0 \\ C - A - 4B = 2 \\ 3B - 6A - 2C = 1 \end{cases}$$

Quicker way of doing this.

The equation has to be true for every x.

In particular

$\underline{x=3}$. $\quad 7 = 10C$

$C = 7/10$.

$\underline{x=1}$: $\quad 3 = -6A$; $\quad A = -\frac{3}{6} = -\frac{1}{2}$.

$x=-2$: $\quad -3 = 15B$; $\quad B = -\frac{3}{15} = -\frac{1}{5}$

$$\int \frac{2x+1}{(x-1)(x+2)(x-3)}\,dx.$$

<u>Step 1</u>

$$\frac{2x+1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}.$$

$$= \frac{A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)}{(x-1)(x+2)(x-3)}.$$

$$2x+1 = A(x+2)(x-3) + B(x-1)(x-3)$$
$$+ C(x-1)(x+2).$$

Conclusion:

$$\int \frac{2x+1}{(x-1)(x+2)(x-3)}\,dx$$

$$= \int \left[-\frac{1}{2(x-1)} - \frac{1}{5(x+2)} + \frac{7}{10(x-3)} \right] dx$$

$$= -\frac{1}{2}\ln|x-1| - \frac{1}{5}\ln|x+2| + \frac{7}{10}\ln|x-3|$$

$$+ C.$$

$$\int \frac{dx}{(x-1)^2(x+3)}.$$

$$\frac{1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$= \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$A(x-1)(x+3) + B(x+3) + C(x-1)^2 = 1.$$

Sub: $x=1$ $\quad 4B = 1$

$x=-3$ $\quad 16C = 1$

$x=0$ $\quad -3A + 3B + C = 1$