Lesson 14 Integration By Partial Fractions - continuation.

Case 1: Simple Factors

$$\frac{1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Case 2: Repeated Factors

$$\frac{1}{(x-1)^2 (x+2)^3 (x+3)} =$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

$$+ \frac{F}{x+3}.$$

Case 3: Irreducible Factors

Example: Compute

$$\int_1^2 \frac{dx}{x(x^2+2)}$$

$$\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$= \frac{A(x^2+2) + x(Bx+C)}{x(x^2+2)}$$

$$1 = A(x^2+2) + x(Bx+C).$$

$$1 = Ax^2 + 2A + Bx^2 + Cx$$

$$\boxed{1 = (A+B)x^2 + Cx + 2A.}$$

$A + B = 0$

$C = 0$

$2A = 1 \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$

<u>Conclusion:</u>

$$\boxed{\frac{1}{x(x^2+2)} = \frac{1}{2x} - \frac{x}{2(x^2+2)}}$$

$$= \frac{x^2+2-x^2}{2x(x^2+2)}$$

$$= \frac{2}{2x(x^2+2)} = \frac{1}{x(x^2+2)}$$

$$\int_1^2 \frac{dx}{x(x^2+2)} = \int_1^2 \left[\frac{1}{2x} - \frac{x}{2(x^2+2)}\right] dx$$

$$= \int_1^2 \frac{dx}{2x} - \frac{1}{2}\int_1^2 \frac{x}{x^2+2}\, dx$$

$x^2+2=u$
$du = 2x\,dx$

$$= \frac{1}{2}\ln|x|\Big|_1^2 - \frac{1}{2}\int_3^6 \frac{du}{2u}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln u\Big|_3^6$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}(\ln 6 - \ln 3)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 = \frac{1}{4}\ln 2.$$

$$\frac{1}{(x-1)^2(x^2+2)^2} =$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

<u>Example:</u>

$$\int_1^2 \frac{dx}{x(x^2+2)^2}.$$

$$\frac{1}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}.$$

$$= \frac{A(x^2+2)^2 + x(x^2+2)(Bx+C) + x(Dx+E)}{x(x^2+2)^2}.$$

$1 = A(x^2+2)^2 + x(x^2+2)(Bx+C) + x(Dx+E).$

Set $x=0$ $\quad 1 = 4A,\ \boxed{A = 1/4}$

Terms in x^3 $\quad Cx^3,\ C=0$

Term in x $\quad (2C+E)x$

$2C+E=0$

$E=0$

① Set $x=1$

$1 = 9A + 3B + D \qquad 3B + D = 1 - 9/4$

$3B+D = -5/4$

$x = -1 \quad 9A$

Term in x^4: $(A+B)x^4$

$A+B=0$, $B=-1/4$.

$3B+D=-5/4$, $-3/4+D=-5/4$

$D=3/4-5/4=-2/4=-1/2$

<u>Conclusion:</u>

$$\frac{1}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

$$=\frac{1}{4x}+\frac{-x}{4(x^2+2)}-\frac{x}{2(x^2+2)^2}.$$

$$\int \frac{dx}{x(x^2+2)^2} =$$

$$= \frac{1}{4}\int \frac{dx}{x} - \frac{1}{4}\underbrace{\int \frac{x\,dx}{x^2+2}} - \frac{1}{2}\underbrace{\int \frac{x\,dx}{(x^2+2)^2}}$$

$$\rightarrow \int \frac{x\,dx}{x^2+2} = \int \frac{du}{2u} = \frac{1}{2}\ln|u|$$

$u = x^2+2$, $du = 2x\,dx$

$$= \frac{1}{2}\ln|x^2+2| + c.$$

$$\rightarrow \int \frac{x\,dx}{(x^2+2)^2} = \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + c$$

$$= -\frac{1}{2(x^2+2)} + c.$$

$$\int \frac{dx}{x(x^2+2)^2} =$$

$$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+2| + \frac{1}{4}\frac{1}{(x^2+2)} + C$$

If E and C were not zero

$$\int \frac{dx}{x^2+2} -$$

$$\int \frac{dx}{(x^2+2)^2} \cdot -$$

$$\int \frac{dx}{(x^2+2)^2}$$

$$x = \sqrt{2}\tan\theta$$

$$dx = \sqrt{2}\sec^2\theta\, d\theta$$

$$= \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{(2\tan^2\theta + 2)^2}$$

$$= \frac{\sqrt{2}}{4}\int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta + 1)^2}$$

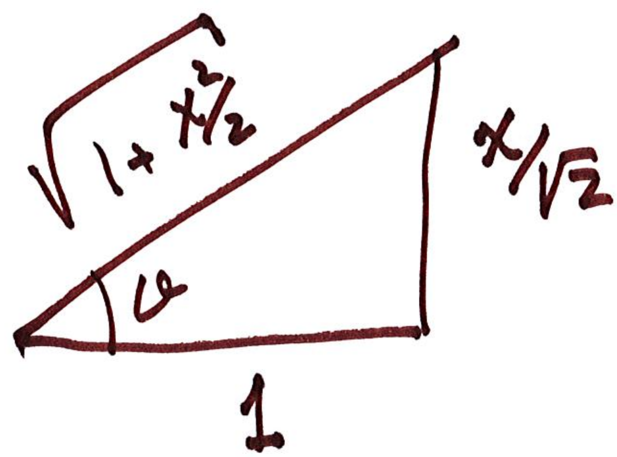
$$= \frac{\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \frac{\sqrt{2}}{4}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta.$$

$$= \frac{\sqrt{2}}{4}\int \frac{1+\cos 2\theta}{2}\, d\theta.$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta \right) + C.$$

$$\tan\theta = x/\sqrt{2} \qquad \theta = \arctan\left(x/\sqrt{2}\right)$$



$$\cos\theta = \frac{1}{\sqrt{1+x^2/2}}, \qquad \sin\theta = \frac{x}{\sqrt{2}\sqrt{1+x^2/2}}$$

$$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \frac{x}{\sqrt{2}\left(1+x^2/2\right)}.$$