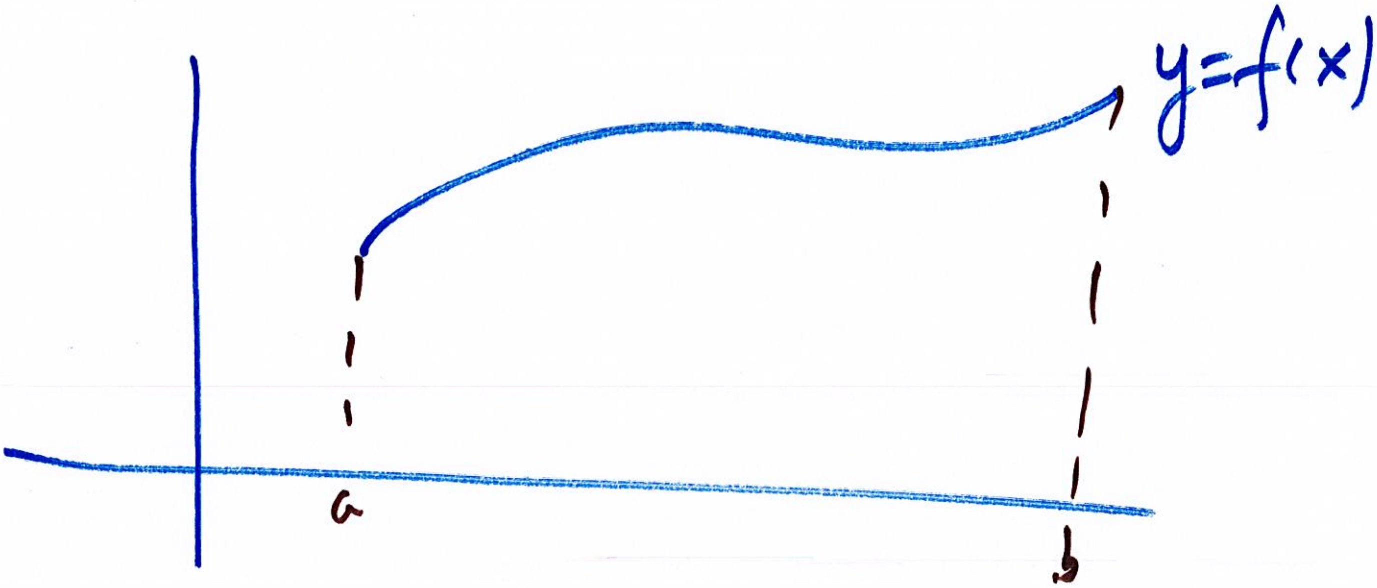
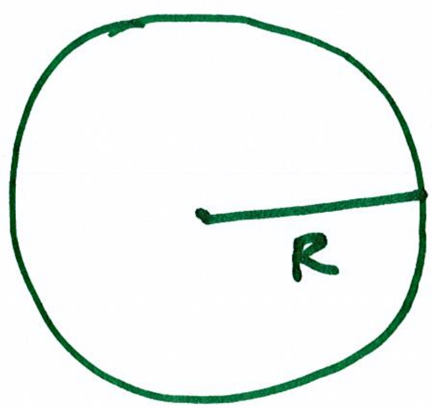
Lesson 17 Sections 8.1 and 8.2

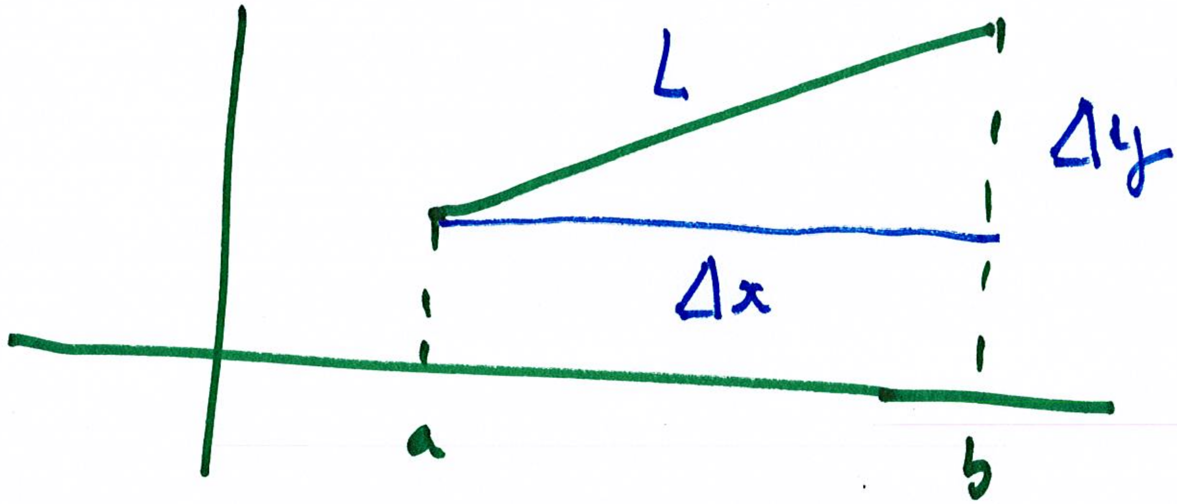
Section 8.1 Arc Length



Example:



$L = 2\pi R$



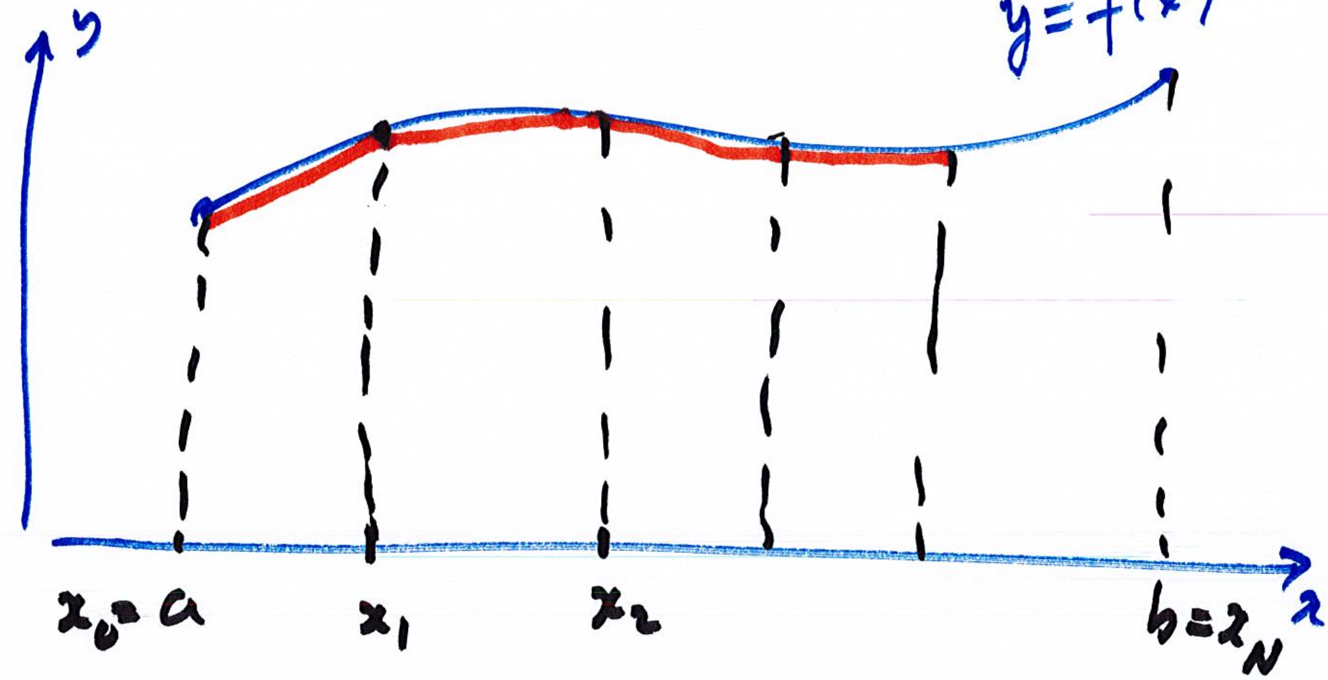
$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right.}.$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \; \Delta x.$$

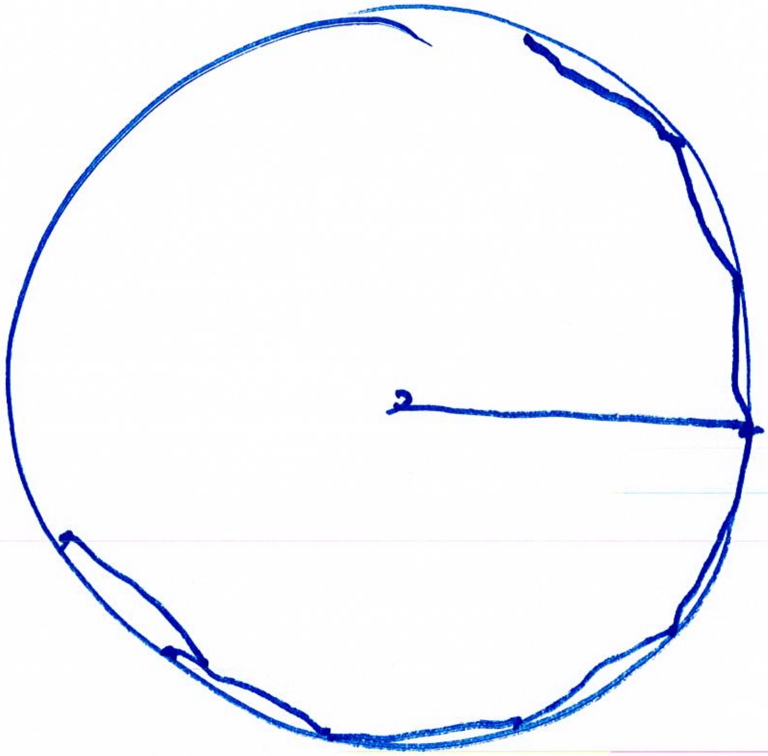
General Case

$f(x)$ and $f'(x)$ are continuous.



Step 1: Divide the interval $[a, b]$ into N equal parts.

Step 2: Length of each of the segments joining $(x_{j-1}, f(x_{j-1}))$ $(x_j, f(x_j))$.



The length of each segment

is $L = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$.

$$\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} = \frac{\Delta y}{\Delta x} = f'(x_j^*) \qquad x_j^* \text{ in } [x_{j-1}, x_j]$$

$$L = \sqrt{1 + (f'(x_j^*))^2}\, \Delta x.$$

__Step 3:__ Add the lengths of the segments

$$\Delta x \left(\sqrt{1+(f'(x_1^*))^2} + \sqrt{1+(f'(x_2^*))^2} + \cdots + \sqrt{1+(f'(x_N^*))^2} \right)$$

__Step 4:__ Take the limit as $N \to \infty$

Length of the curve

$$= \int_a^b \sqrt{1+(f'(x))^2}\, dx.$$

Find the length of

the curve

$$y = 1 + x^{3/2} \quad 0 \le x \le 1$$

$$f(x) = 1 + x^{3/2}$$

$$L = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{3}{2} x^{1/2} \ , \ (f'(x))^2 = \frac{9}{4} x$$

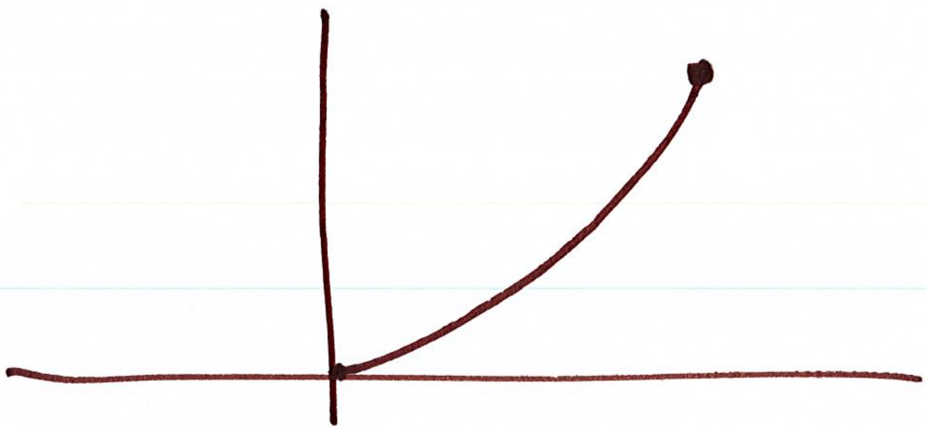
$$L = \int_0^1 \sqrt{1 + \frac{9}{4} x}\, dx$$

$u = 1 + \frac{9}{4} x \quad du = \frac{9}{4} dx$

$$= \frac{4}{9}\int_{1}^{13/4} u^{1/2}\,du = \frac{4}{9}\cdot\frac{2}{3}\,u^{3/2}\Big|_{1}^{13/4}$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right).$$

$$y = \frac{1}{2} x^2 , \quad 0 \le x \le 1$$



$$f(x) = \frac{1}{2} x^2 ; \quad f'(x) = x$$

$$L = \int_0^1 \sqrt{1 + (f'(x))^2} \, dx$$

$$= \int_0^1 \sqrt{1 + x^2} \, dx .$$

$x = \tan\theta$, $dx = \sec^2\theta \, d\theta$

$$= \int_0^{\pi/4} \sec\theta . \sec^2\theta \, d\theta .$$

$$= \int_0^{\pi/4} \sec^3\theta \, d\theta.$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta$$
$$+ \frac{1}{2} \ln|\sec\theta + \tan\theta| + C.$$

$$\int_0^{\pi/4} \sec^3\theta \, d\theta = \frac{1}{2} \cdot \sqrt{2} + \frac{1}{2} \ln|1 + \sqrt{2}|$$

$$y = \frac{x^3}{3} + \frac{1}{4x} \qquad 1 \le x \le 2.$$

$$f(x) = x^3/3 + 1/4x$$

$$f'(x) = x^2 - \frac{1}{4x^2}$$

$$L = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

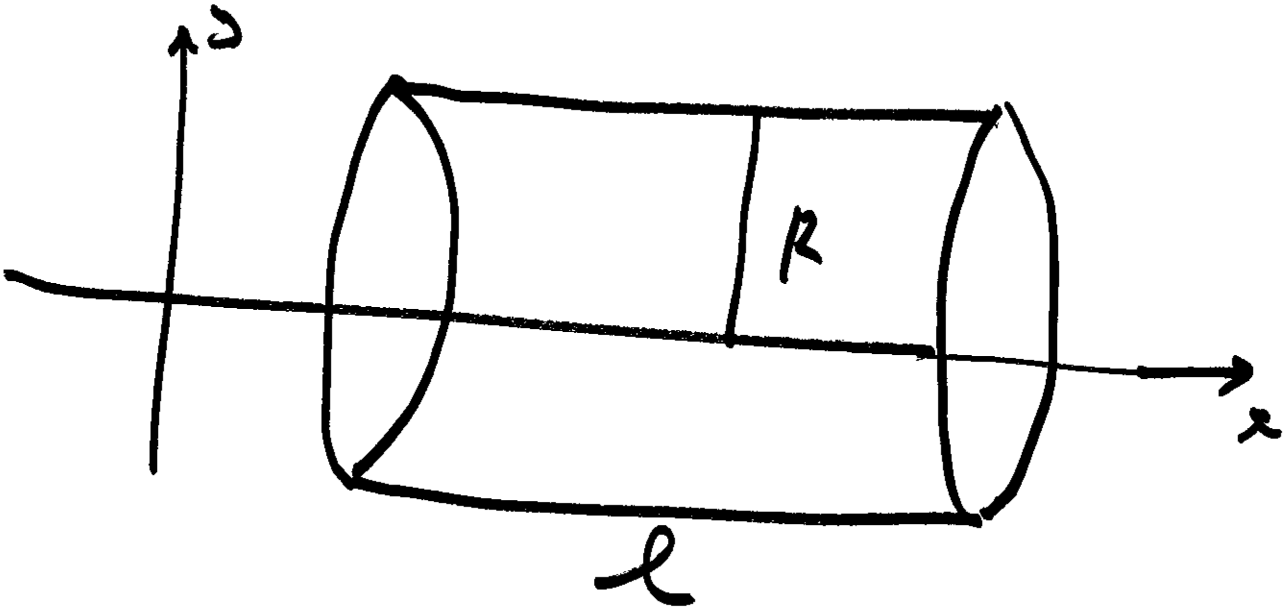
$$1 + (f'(x))^2 = 1 + \left(x^2 - \frac{1}{4x^2}\right)^2$$

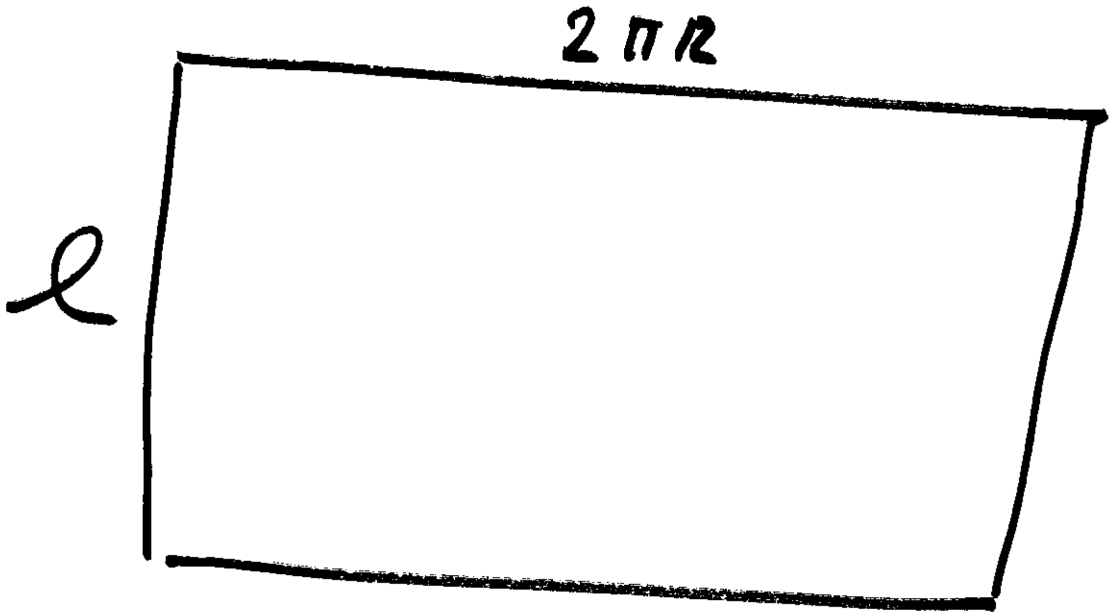
$$= 1 + x^4 - 1/2 + 1/16x^4$$

$$= x^4 + 1/2 + 1/16x^4 = \left(x^2 + 1/4x^2\right)^2$$

$$\sqrt{1 + (f'(x))^2} = x^2 + \frac{1}{4x^2}.$$

Area of the Surface of



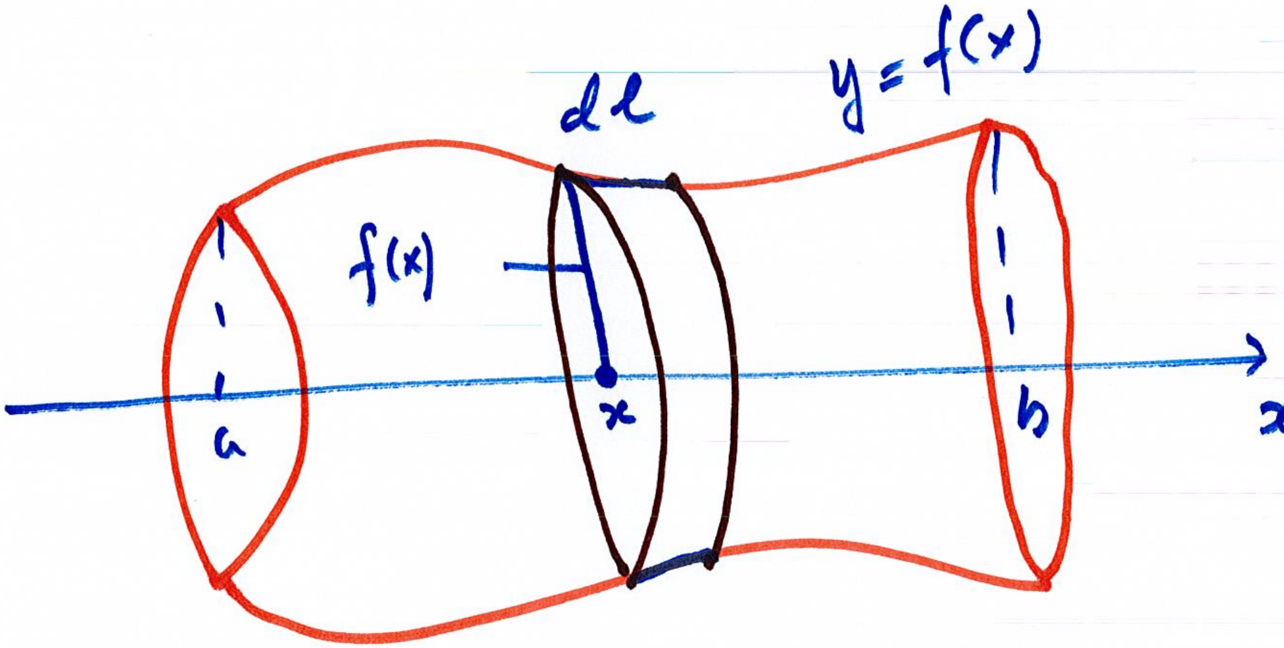


Area = $2\pi R l$.

$$L = \int_1^2 \left(x^2 + \frac{1}{4x^2} \right) dx$$

$$= \left(\frac{x^3}{3} - \frac{1}{4x} \right) \Big|_1^2 = \ldots$$

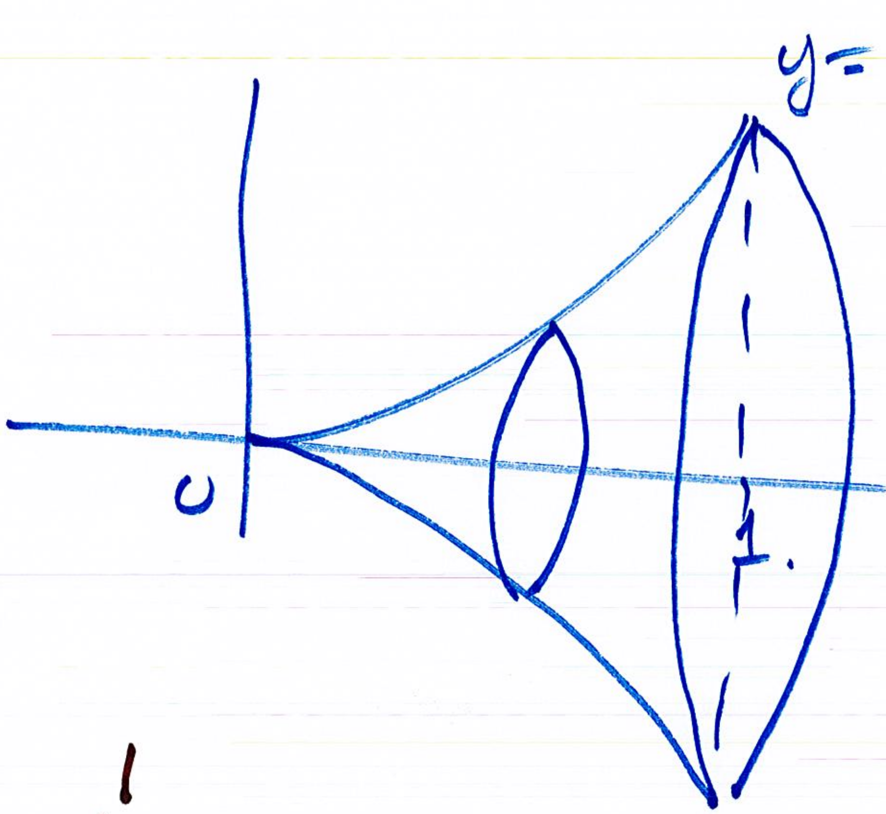
Section 8.2 Areas of Surfaces of revolution.



$$dA = 2\pi \underbrace{f(x)}_{\text{Radius}} \cdot \underbrace{\sqrt{1+(f'(x))^2}\, dx}_{d\ell}$$

~~Volume~~ Area $= 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$

Example.



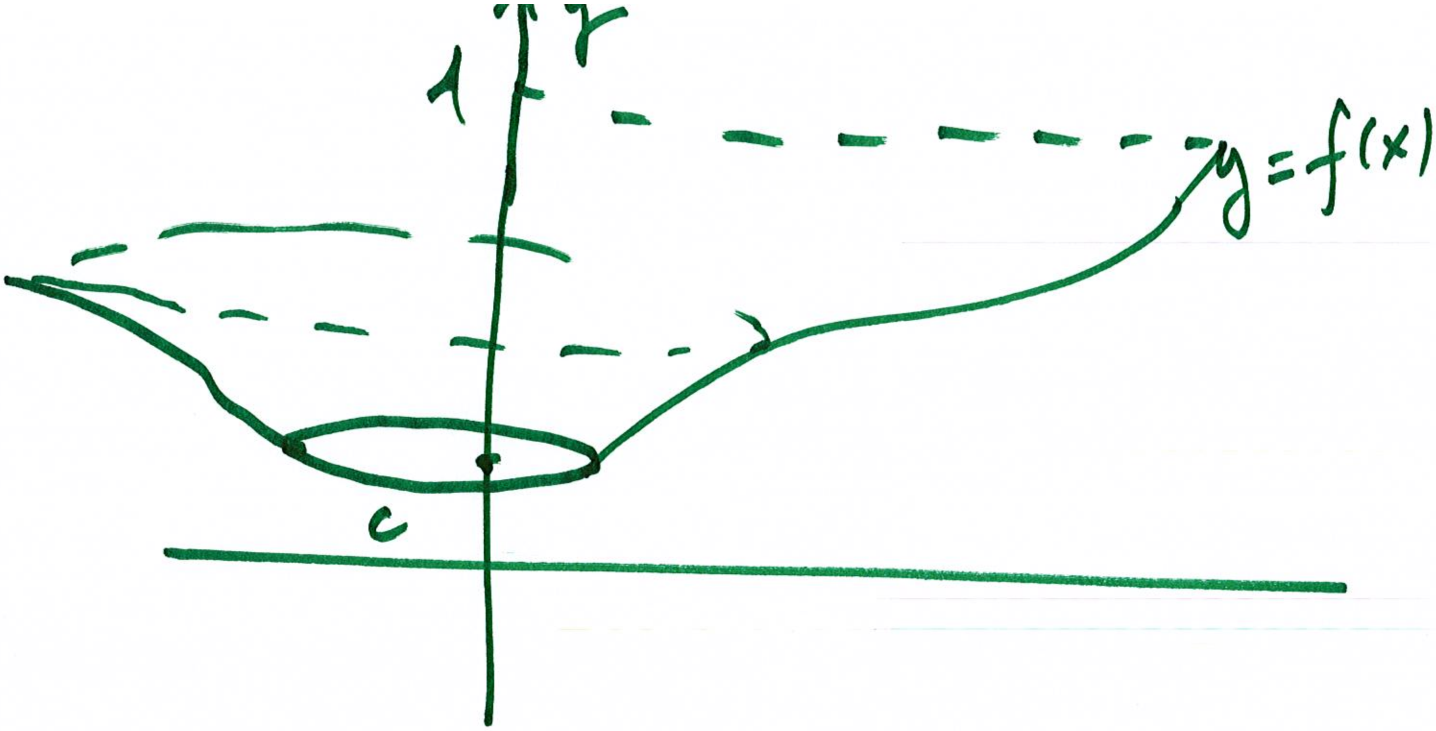
$f(x) = x^3$

$f'(x) = 3x^2$

$$\text{Area} = 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\,dx$$

$u = 1+9x^4, \quad du = 36x^3\,dx$

$$\text{Area} = \frac{2\pi}{36} \int_1^{10} u^{1/2}\,du.$$

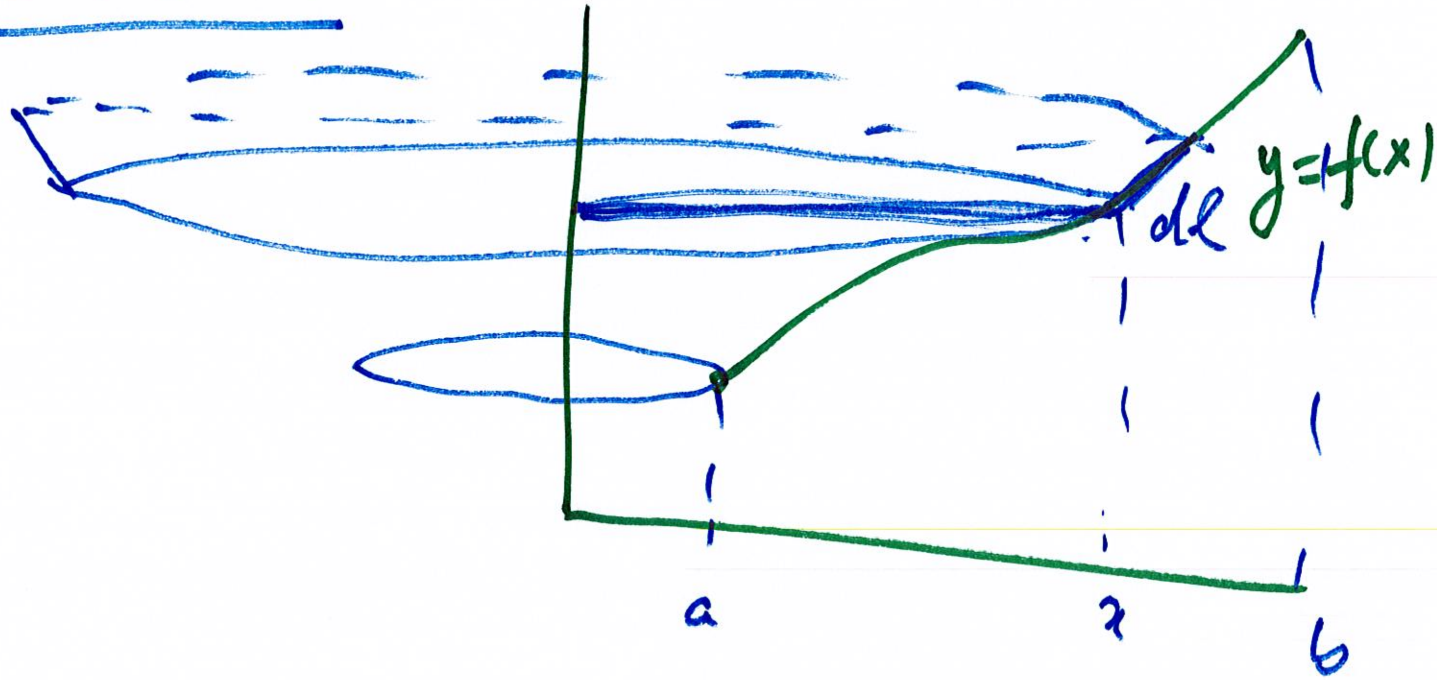


Q.

Solution 1: Write $x = g(y)$

$$A = 2\pi \int_c^d g(y)\sqrt{1+(g'(y))^2}\,dx$$

Solution 2



$$dA = 2\pi x \sqrt{1+(f'(x))^2}\, dx$$

$$A = 2\pi \int_a^b x \sqrt{1+(f'(x))^2}\, dx$$

Example: $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$

$1 \le x \le 2$

Rotate about the y axis.

$$A = \underline{2\pi} \int_1^2 x \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \tfrac{1}{2}x - \tfrac{1}{2x}$$

$$(f'(x))^2 = \tfrac{1}{4}x^2 - \tfrac{1}{2} + \tfrac{1}{4x^2}$$

$$1 + (f'(x))^2 = \tfrac{1}{4}x^2 + \tfrac{1}{2} + \tfrac{1}{4x^2}$$

$$= \left(\tfrac{1}{2}x + \tfrac{1}{2x}\right)^2$$

$$A = \int_1^2 x \left(\tfrac{1}{2}x + \tfrac{1}{2x}\right) dx.$$