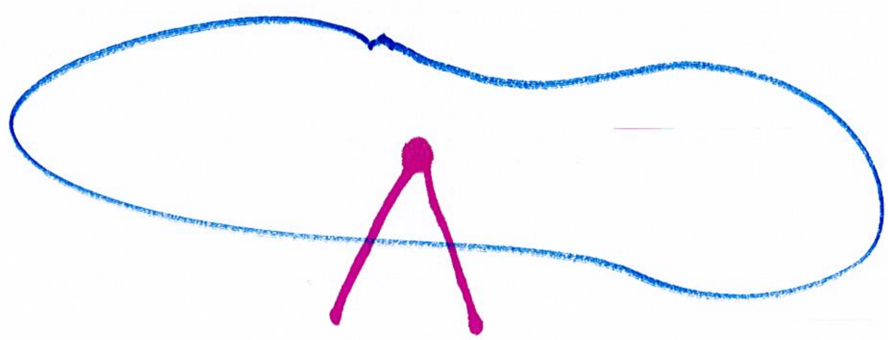
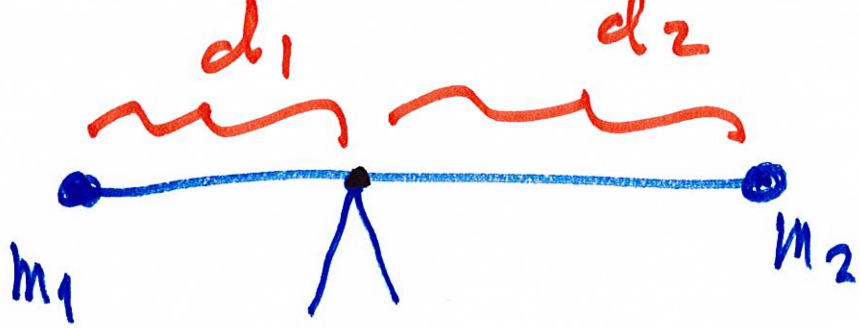
Lesson 18 Section 8.3

Center of Mass

Goal: Find the center of mass of a thin 2-dimentional object (laminas)

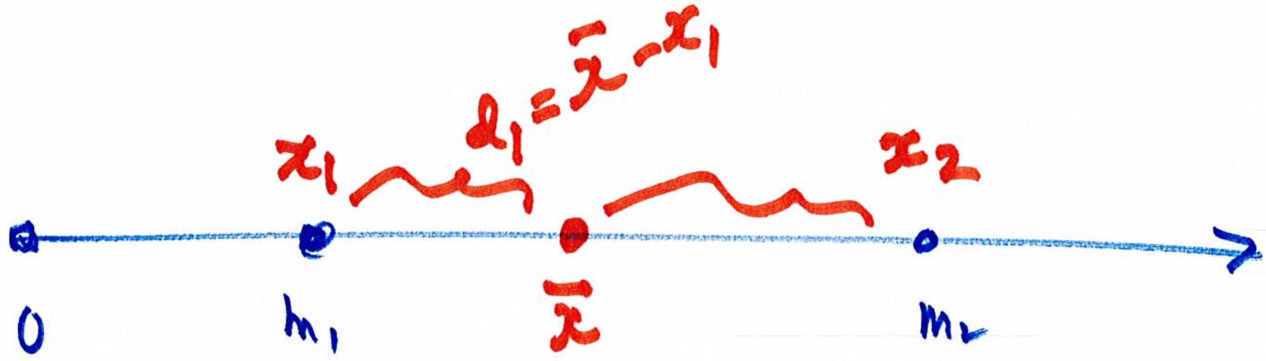


We will assume the object is homogeneous. = Constant density of Mass



moment of m_1 $= m_1 d_1 = \underbrace{m_2 d_2}_{} =$ moment of m_2

Find the center of mass in coordinates

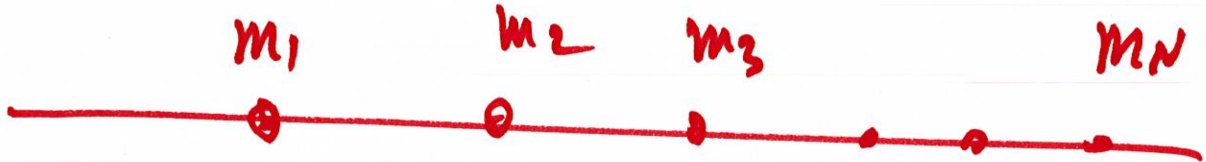


$$m_1 \cdot (\bar{x} - x_1) = m_2 (x_2 - \bar{x})$$

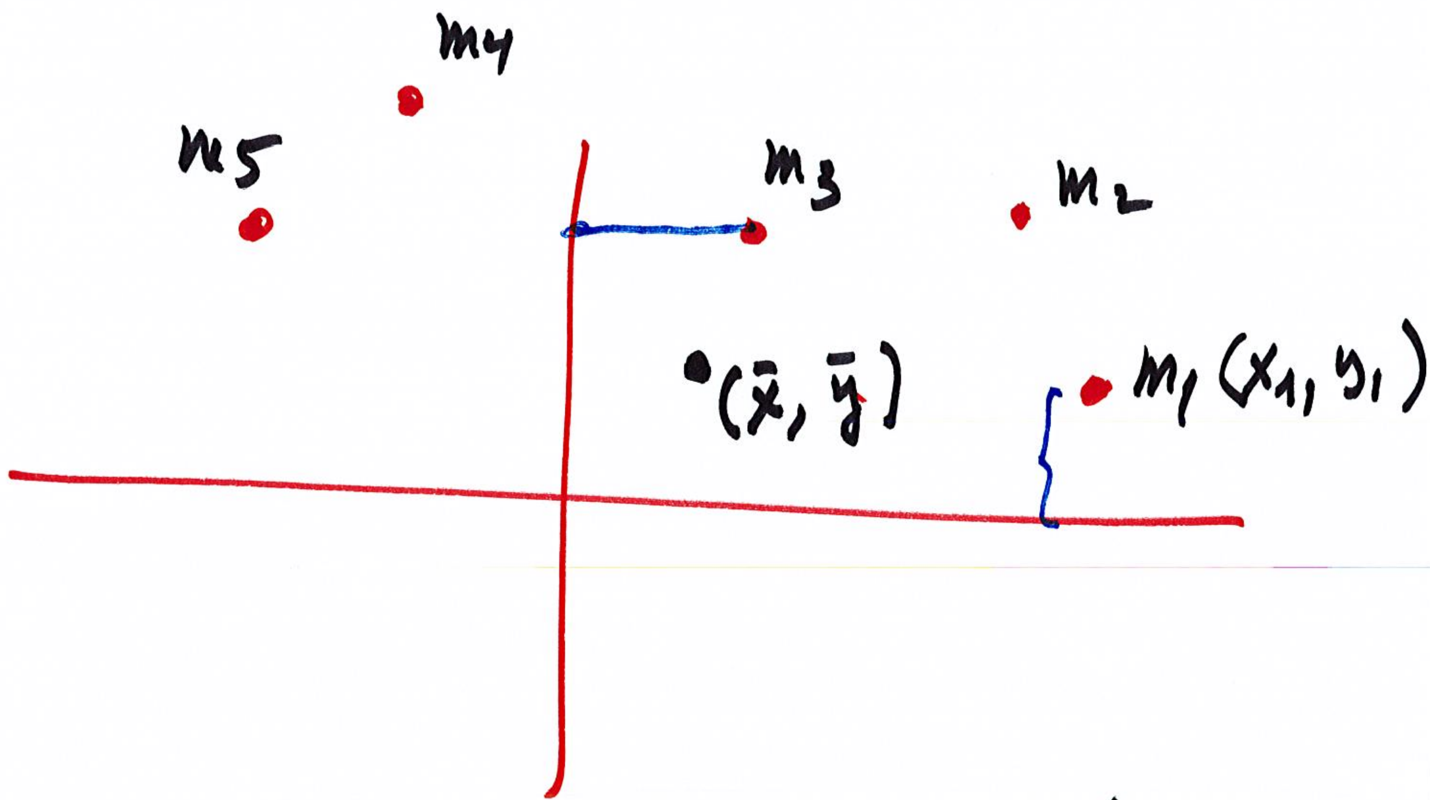
$$m_1 \bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$



$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N}}$$



Moment of m_1 with respect to the $\underline{\underline{x}}$-axis

$= m_1 y_1$

Total moment with respect to the x-axis

$$M_x = m_1 y_1 + m_2 y_2 + \dots + m_N y_N$$

~~Momen~~ Moment of

$m_j(x_j, y_j)$ with respect

the y-axis is

~~$M_j y_j$~~ $m_j x_j$

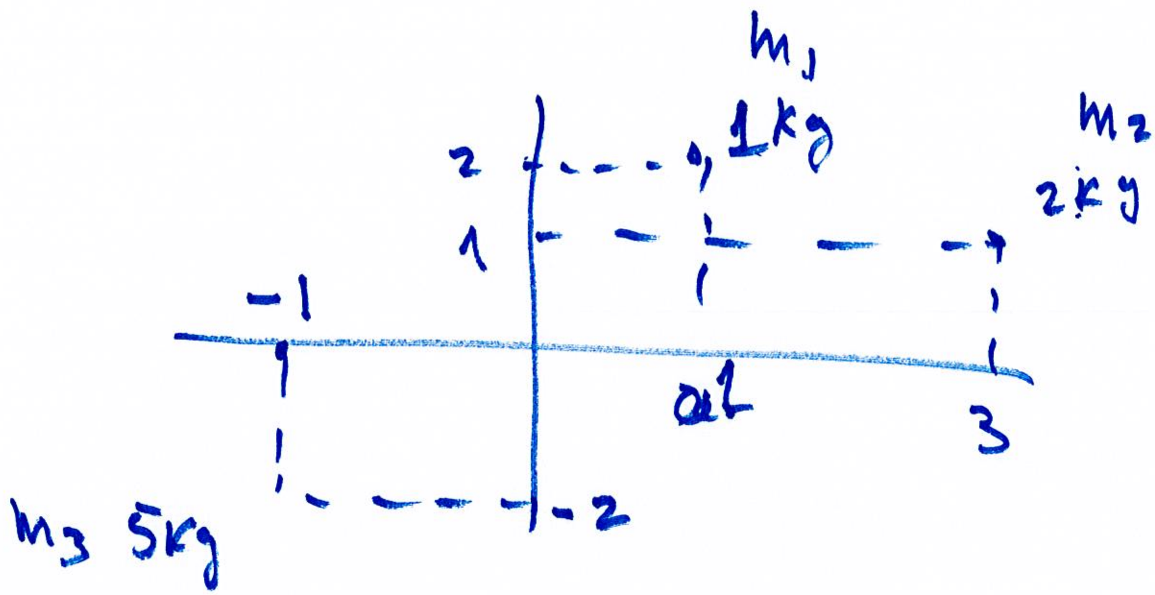
$$M_y = m_1 x_1 + m_2 x_2 + \cdots + m_N x_N.$$

$$\bar{x} = \frac{M_y}{M} \; ; \; \bar{y} = \frac{M_x}{M}.$$

Are the coordinates of the

Center of mass

$$M = m_1 + m_2 + m_3 + \cdots + m_N.$$

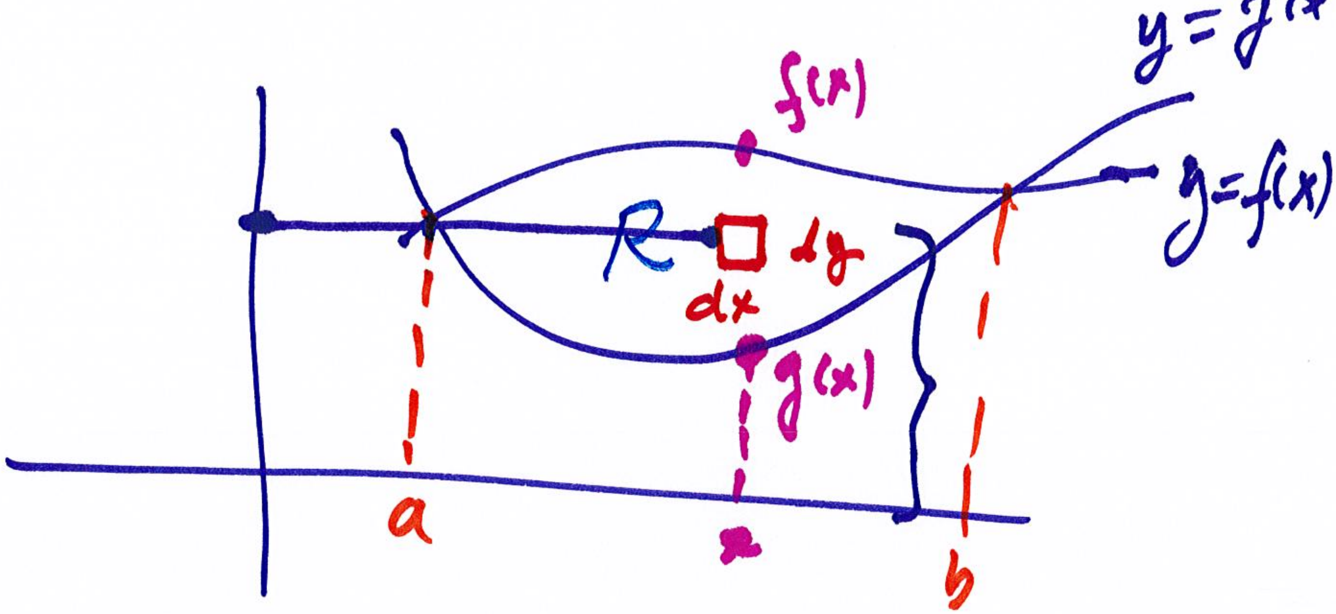


$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1 + 6 - 5}{8}$$

$$= 2/8$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{2 + 2 - 10}{8}$$

$$= -6/8$$



ρ = density of mass is constant

$= \text{mass}/\text{Area}.$

$$\boxed{m = \rho \, dx \, \underline{\underline{dy}}}$$

Total mass

$$M = \int_a^b \left(\int_{g(x)}^{f(x)} \rho \, dy \right) \underline{\underline{dx}}$$

$$= \rho \int_a^b (f(x) - g(x)) \, dx.$$

Moment with respect to the y-axis

$$dM_y = \rho \underline{\underline{x}}\, dx\, dy$$

$$M_y = \int_a^b \rho\, x \left(\int_{g(x)}^{f(x)} dy \right) dx$$

$$\boxed{M_y = \rho \int_a^b x \left(f(x) - g(x) \right) dx}$$

$$\boxed{\bar{x} = \frac{M_y}{M}}$$

Moment with respect to the x-axis

$$dM_x = \rho\, y\, dx\, dy$$

$$M_x = \rho \int_a^b \left(\int_{g(x)}^{f(x)} y\, dy \right) dx$$

$$M_x = \rho \int_a^b \frac{(f(x))^2 - (g(x))^2}{2}\, dx$$

$$= \frac{\rho}{2} \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx.$$

$$\boxed{\bar{y} = \frac{M_x}{M}}$$

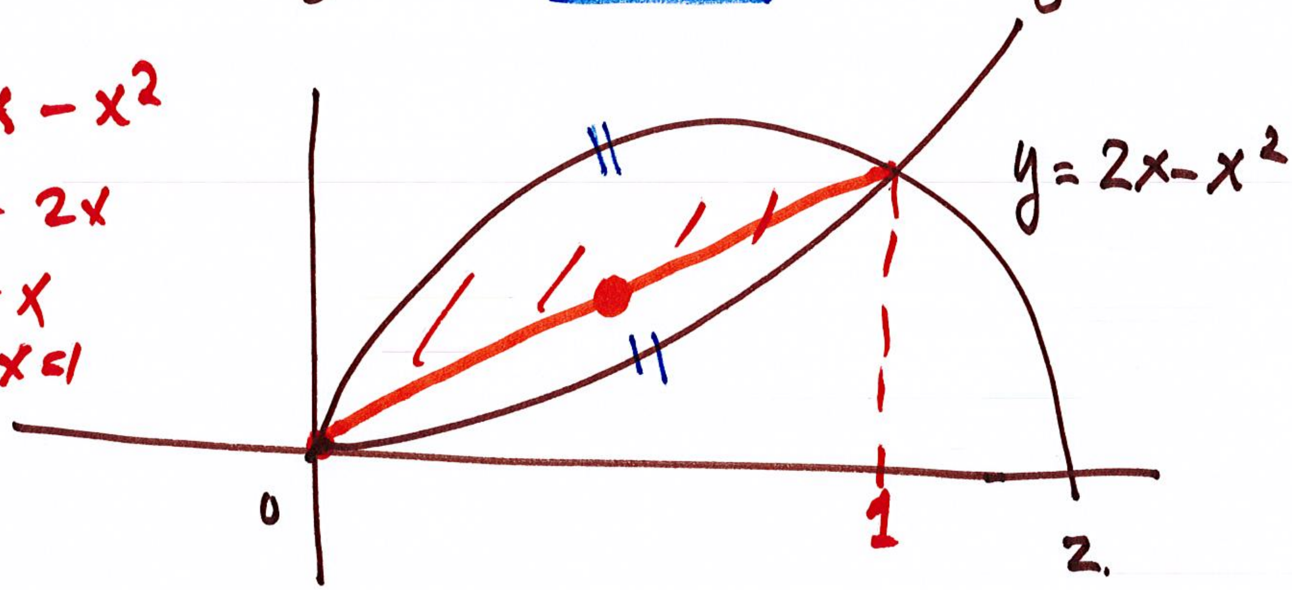
Example. Find the center of mass (Centroid) of the region $\rho = 1$

$x^2 = 2x - x^2$

$2x^2 = 2x$

$x^2 = x$

$x = 0, \; x = 1$



Step 1: Find the area.

$$A = \int_0^1 (2x - x^2 - x^2)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx = 2\left(\tfrac{1}{2} - \tfrac{1}{3}\right)$$

$$= 2 \cdot \tfrac{1}{6} = \tfrac{1}{3}.$$

$$M_x = \frac{1}{2}\int_a^b \left((f(x))^2 - (g(x))^2 \right) dx$$

$$f(x) = \underline{2x - x^2}, \quad g(x) = x^2$$

$$a = 0, \quad b = 1$$

$$M_x = \frac{1}{2}\int_0^1 (4x^2 - 4x^3 + x^4 - x^4)\, dx$$

$$= \frac{1}{2}\int_0^1 (4x^2 - 4x^3)\, dx$$

$$= 2.\left(\frac{1}{3} - \frac{1}{4}\right) = 2 \cdot \frac{1}{12} = 1/6 .$$

$$\bar{y} = \frac{M_x}{M} = \frac{1/6}{1/3} = 1/2 .$$

$$M_y = \int_a^b x(f(x) - g(x))\,dx$$

$$M_y = \int_0^1 x(2x - 2x^2)\,dx$$

$$= \int_0^1 (2x^2 - 2x^3)\,dx$$

$$= 2 \cdot \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{1}{6}.$$

$$\boxed{\cancel{M_y} = \cancel{1/2}}$$

$$\bar{y} = \frac{M_y}{M} = \frac{1/6}{1/3} = 1/2$$