Lesson 20

Sequences and Series

Exam 2, Friday 10/19

Covers Lessons 11 to 20

Sequences

$$\lim_{n\to\infty} \frac{n^4+4n^3+3n+1}{10n^4+8n^2+1} = ?$$

Solution 1: $\dfrac{n^4+4n^3+3n+1}{10n^4+8n^2+1} = \dfrac{\cancel{n^4}\left(1+\frac{4}{n}+\frac{3}{n^3}+\frac{1}{n^4}\right)}{\cancel{n^4}\left(10+\frac{8}{n^2}+\frac{1}{n^4}\right)}$

$$= \frac{1+4/n+\frac{3}{n^3}+\frac{1}{n^4}}{10+8/n^2+1/n^4}$$

$$\lim_{n\to\infty} \frac{n^4+4n^3+3n+1}{10n^4+8n^2+1} =$$

$$= \lim_{n\to\infty} \frac{1+4/n+3/n^3+1/n^4}{10+8/n^2+1/n^4}$$

$$= \frac{1}{10}$$

$$\lim_{n\to\infty} \frac{7n^3+9n+2}{10n^2+5n+1}.$$

$$= \lim_{n\to\infty} \frac{n^3\left(7+9/n^2+2/n^3\right)}{n^2\left(10+5/n+1/n^2\right)}$$

$$= \lim_{n\to\infty} \underline{\underline{n}} \; \frac{7+9/n^2+2/n^3}{10+5/n+1/n^2} = \infty.$$

$$\lim_{n\to\infty} \frac{8n^3 + 10n^2 + 5}{n^4 + 50n + 1}$$

$$= \lim_{n\to\infty} \frac{n^3\left(8 + 10/n + 5/n^3\right)}{n^4\left(1 + 50/n^3 + 1/n^4\right)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \; \frac{8 + 10/n + 5/n^3}{1 + 50/n^3 + 1/n^4}$$

$$= 0.$$

Solution 2 :

$$\lim_{n\to\infty} \frac{8n^3 + 10n + 5}{7n^3 + 8n + 1}$$

$$f(x) = \frac{8x^3 + 10x + 5}{7x^3 + 8x + 1}$$

If $\boxed{\lim_{x \to \infty} f(x) = L}$ I

Then

$\boxed{\lim_{n \to \infty} f(n) = L.}$ II

II is a special case of I.

$$\lim_{x \to \infty} \frac{8x^3 + 10x + 5}{7x^3 + 8x + 1}$$

$$= \lim_{x \to \infty} \frac{24x^2 + 10}{21x^2 + 8}$$

$$= \lim_{x \to \infty} \frac{48 \cancel{x}}{42 \cancel{x}} = \frac{48}{42} = 8/7.$$

Ex: $\lim_{n \to \infty} n \sin(1/n)$.

$$\lim_{x \to \infty} x \sin(1/x)$$

$$x \sin\left(\frac{1}{x}\right) = \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = 1.$$

$$\boxed{\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = 1}$$

$$\boxed{\frac{1}{3} = 0,33333\ldots}$$

$$= (0.3) + (0.03) + (0.003) + \ldots$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$$

$$a_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$$

$$a_1 = \frac{3}{10}, \quad a_2 = \frac{3}{10} + \frac{3}{10^2}$$

$$a_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} +$$

$$\boxed{\frac{1}{3} = \lim_{n\to\infty} a_n .}$$

$$a_n = \underline{\underline{\frac{3}{10}}} + \frac{3}{\underbrace{\cancel{10^2}}} + \frac{3}{\cancel{10^3}} + \cdots + \frac{3}{\cancel{10^n}}$$

<u>Idea</u>:

$$\frac{1}{10} a_n = \frac{3}{\underbrace{\cancel{10^2}}} + \frac{3}{\underbrace{\cancel{10^3}}} + \frac{3}{10^4} + \cdots + \frac{3}{\cancel{10^n}} + \underline{\underline{\frac{3}{10^{n+1}}}}$$

$$a_n - \frac{1}{10} a_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\left(1 - \frac{1}{10}\right) a_n = \frac{3}{10} - \frac{3}{10^{n+1}} .$$

$$9/10 \; a_n = \frac{3}{10} - 3/10^{n+1} .$$

$$a_n = \frac{3/10 - \frac{3}{10^{n+1}}}{9/10}$$

$\frac{3}{10^{n+1}} \to 0$

$$\lim_{n\to\infty} a_n = \frac{3/10}{9/10}$$

$$= 3/9 = 1/3.$$

Geometric Series

$$S_n = 1 + r + r^2 + r^3 + \cdots + r^n$$

Ex: $r = 1/10$

$$S_n = 1 + 1/10 + 1/10^2 + \frac{1}{10^3} + \cdots + \frac{1}{10^n}.$$

$$\lim_{n\to\infty} S_n = ?$$

$$\lim_{n\to\infty} r^n = ?$$

If $\underline{r > 1}$

$\underline{r=2}$: 2^n $\quad r^n \to \infty$ if $r > 1$

If $r < -1$

$\underline{r=-2}$ $\quad (-2)^n \begin{cases} \to -\infty & \text{if } n \text{ odd} \\ \to \infty & \text{if } n \text{ is even} \end{cases}$

$\lim_{n\to\infty} r^n$ does not exist if $r < -1$.

$\underline{r=1}$: $r^n = 1$

$S_n = 1 + r + r^2 + r^3 + \cdots + r^n$

$S_n = 1 + 1 + 1 + 1 + 1 + 1 + \cdots$

$= n+1 \to \infty$

$\underline{r=-1}$: $(-1)^n = \begin{cases} 1, & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

when $|r| < 1$

$$\lim_{n \to \infty} r^{n+1} = 0$$

$$\lim_{n \to \infty} S_n = ?$$

$$S_n = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \dots + \cancel{r^n}$$

$$r S_n = \cancel{r} + \cancel{r^2} + \cancel{r^3} + r^4 + \dots + \cancel{r^n} + r^{n+1}$$

$$S_n - r S_n = 1 - r^{n+1}$$

$$S_n (1 - r) = 1 - r^{n+1}$$

$$\boxed{S_n = \frac{1 - r^{n+1}}{1 - r}}$$

Conclusion:

$$S_n = 1 + r + r^2 + \dots + r^n.$$

$$\boxed{\lim_{n\to\infty} S_n = \frac{1}{1-r} \quad \text{if } |r| < 1}$$

In general.

We start with

a sequence a_n, $n = 1, 2, \dots$

In this particular case

$$a_n = r^n.$$

$$S_N = a_1 + a_2 + \dots + a_N$$

$$= \sum_{n=1}^{N} a_n$$

The sequence of partial

sums.