Lesson 21 Sections 11.2 and 11.3

Sequences and Series

We start with a sequence

$a_1, a_2, a_3, \ldots$ $\{a_n\}$

We build a new sequence

$s_1 = a_1$

$s_2 = a_1 + a_2$

$\vdots$

$s_N = a_1 + a_2 + \cdots + a_N$.

s_N is the sequence of partial sums of $\{a_n\}$.

Examples:

$a_1 = r, \quad a_2 = r^2, \quad a_3 = r^3, \ldots$

$a_n = r^n.$

$S_1 = r$

$S_2 = r + r^2$

$S_3 = r + r^2 + r^3$

$\vdots$

$S_N = r + r^2 + \cdots + r^N.$

Question: Does S_N converse?

$$S_N = a_1 + a_2 + \dots + a_N = \sum_{n=1}^{N} a_n$$

$$\lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n.$$

Is the series $\sum_{n=1}^{\infty} a_n$.

Question: Does $\sum_{n=1}^{\infty} a_n$ converge?

Find ways of answering this question.

$$\sum_{n=1}^{N} r^n = r + r^2 + \cdots + r^N$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

provided $|r| < 1$.

Zeroth test of Divergence

$$S_N = \sum_{n=1}^{N} a_n \, , \quad S_N = \sum_{n=1}^{N-1} a_n$$

$$\boxed{S_N - S_{N-1} = a_N.}$$

If $\lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n = L$ converges.

$$\lim_{N\to\infty}(S_N - S_{N-1}) = L - L = 0$$

$$\boxed{\lim_{N\to\infty} a_N = 0}$$

<u>Conclusion:</u> If

$\displaystyle\sum_{n=1}^{\infty} a_n$ converges

then $\displaystyle\lim_{n\to\infty} a_n = 0$

If $\displaystyle\lim_{n\to\infty} a_n$ either does not exist

or $\displaystyle\lim_{n\to\infty} a_n \neq 0$ $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges

$$\sum_{n=1}^{\infty} \frac{n+5}{n+8} \quad \text{diverges.}$$

$$a_n = \frac{n+5}{n+8}, \quad \lim_{n\to\infty} a_n = 1.$$

$$\sum_{n=1}^{\infty} \cos\left(1/n\right) \quad \text{diverges.}$$

$$\lim_{n\to\infty} \cos\left(1/n\right) = 1.$$

Warning!

If $\lim_{n \to \infty} a_n = 0$

one cannot conclude

that $\sum_{n=1}^{\infty} a_n$ converges.

Test of Convergence

The Integral test

Question D.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

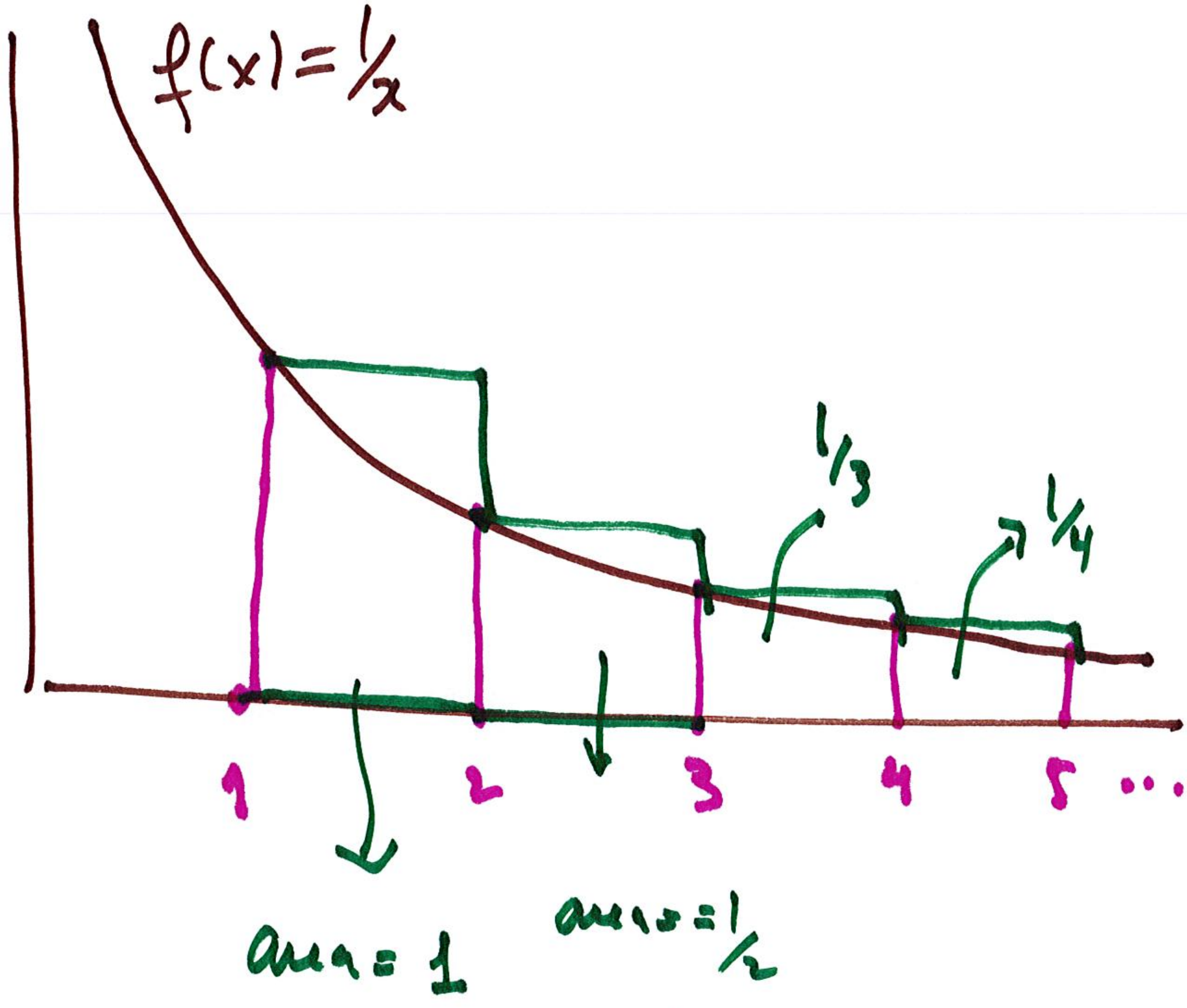
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$$

Converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad a_n = \frac{1}{n}, \quad \lim_{n \to \infty} a_n = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad a_n = \frac{1}{n^2} \quad \lim_{n \to \infty} a_n = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

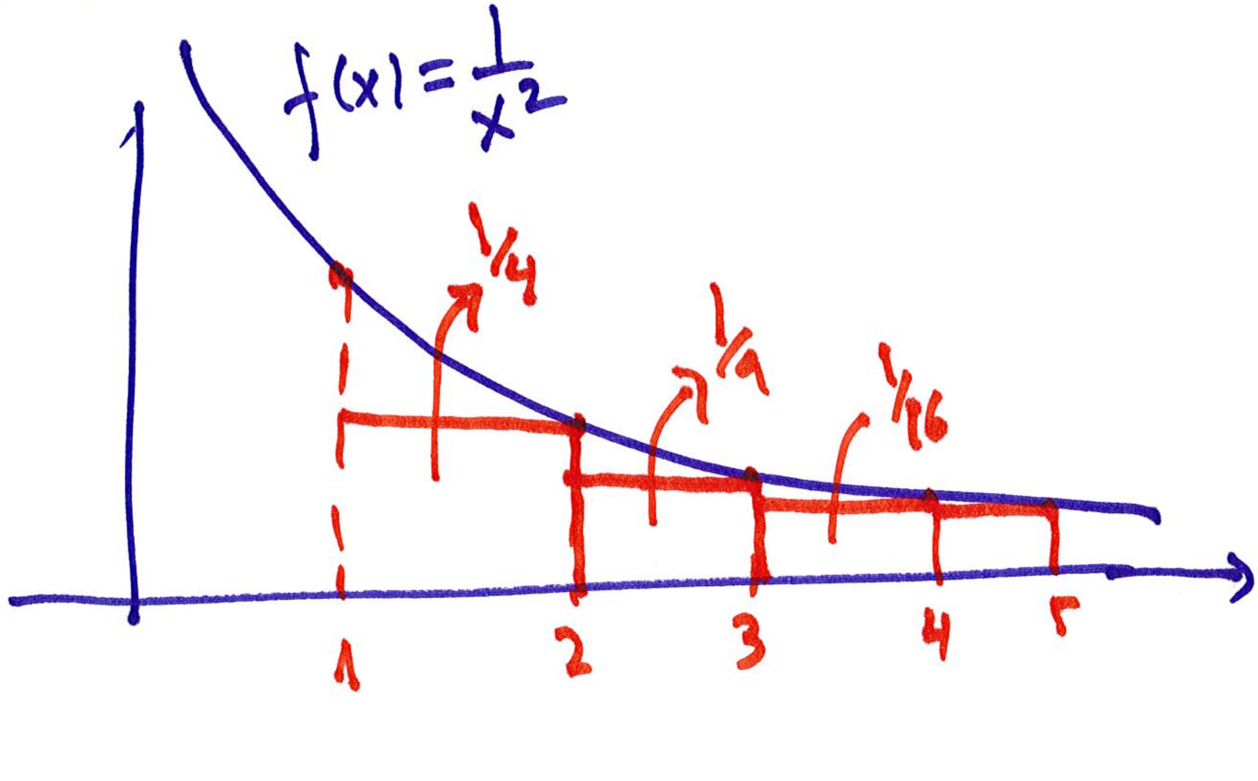


$$1 + \frac{1}{2} + \frac{1}{3} + \cdots > \int_1^{\infty} \frac{1}{x}\, dx.$$

$$\lim_{N\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{N}=\infty$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges.}}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$



Conclusion.

$$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots \leq \int_1^\infty \frac{1}{x^2}\,dx$$

$$0 < \sum_{n=2}^{N} \frac{1}{n^2} \leq \int_1^\infty \frac{1}{x^2}\,dx$$

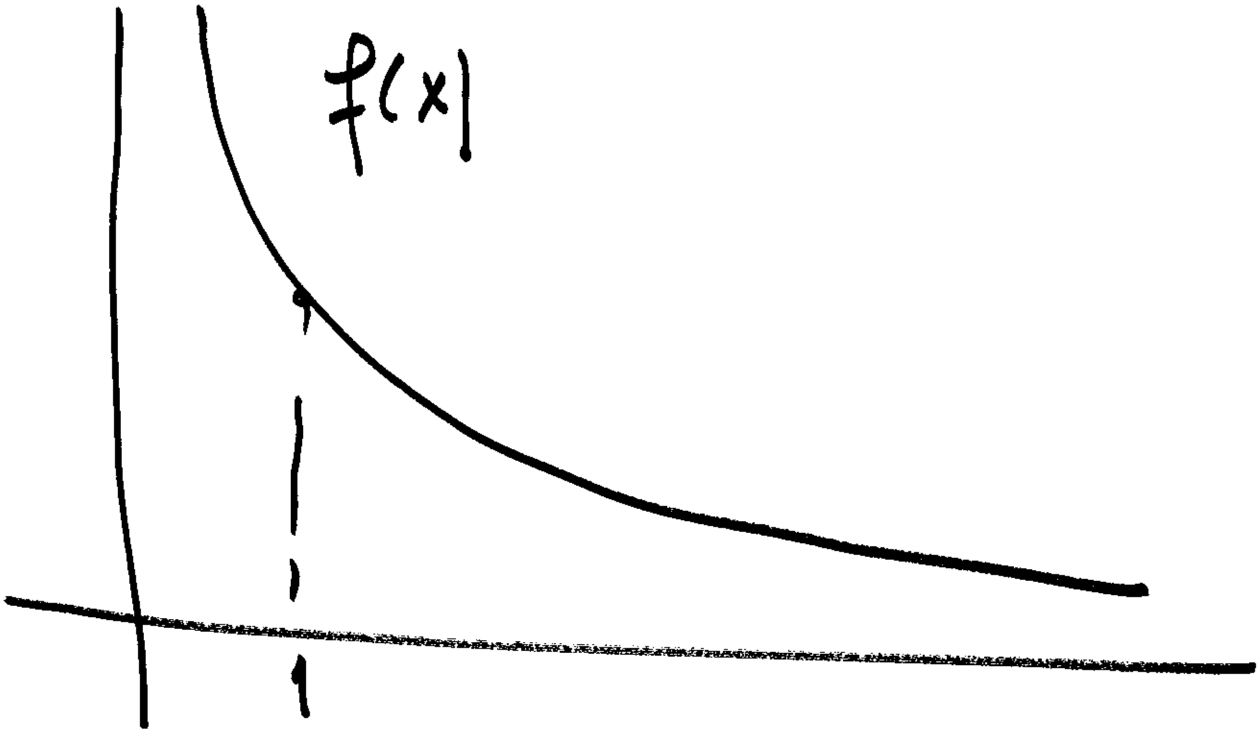
$$S_1 = \frac{1}{4}$$

$$S_2 = \frac{1}{4}+\frac{1}{9}$$

$$S_3 = \frac{1}{4}+\frac{1}{9}+\frac{1}{16}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$ Converges.

The Integral Test



1) $f(x) > 0$, $x \geq 1$

2) $f(x)$ is decreasing.

3) $\lim\limits_{x \to \infty} f(x) = 0$.

$$\sum_{n=1}^{\infty} f(n)$$

Converges if $\int_1^{\infty} f(x)\,dx$ converges

Diverges if $\int_1^{\infty} f(x)\,dx$ diverges.

If $\sum_{n=1}^{\infty} f(n)$ converges

$\int_1^{\infty} f(x)\,dx$ converges

If $\sum_{n=1}^{\infty} f(n)$ diverges, $\int_1^{\infty} f(x)\,dx$ diverges.

For what values of p

Does

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p>0$$

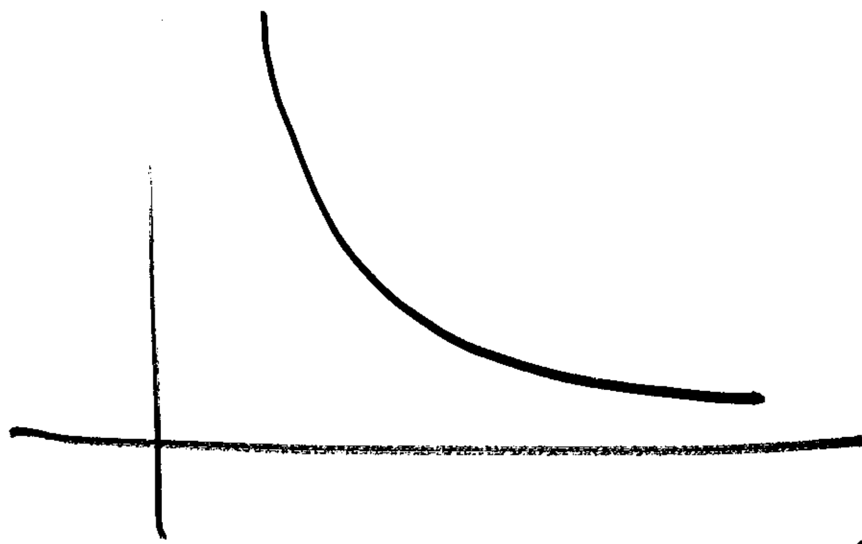
Converge.

$\underline{p=1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$p=2$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Apply the Integral test

$$f(x) = \frac{1}{x^p} \quad , \; p > 0$$



$$\int_1^\infty \frac{1}{x^p}\,dx \quad , \quad \text{Converges if } p > 1$$

diverges if $p \le 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ converges if $p > 1$

diverges otherwise

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \quad p = 1/2$$

diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}, \quad \pi = 3.1\ldots > 1$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad p>0$$

Can we apply the integral test?

$$f(x) = \frac{1}{x(\ln x)^p} \qquad x \geq 2$$

$$f(x) = x^{-1}(\ln x)^{-p}$$

$$f'(x) = -\frac{1}{x^2}(\ln x)^{-p} - p\,x^{-2}(\ln x)^{-p-1}$$

$$\leq 0.$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$$ diverges

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$$ converges.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$$

$u = \ln x, \quad du = \frac{1}{x}\,dx$

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^{\infty} \frac{du}{u^p}$$

Converges if $p > 1$

diverges if $p \le 1$.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

Converges if $p > 1$

Diverges if $p \le 1$