Lesson 22 Section 11.4

The Comparison tests

We want to compare two series.

$$\sum_{n=1}^{\infty} r^n ,$$ Converges if $|r|<1$

Diverges if $|r| \geq 1$

$$\left\{ \begin{array}{ll} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}, & \text{Converges if } p>1 \\ & \text{diverges if } p \leq 1 \\ \displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, & \text{Converges if } p>1 \\ & \text{diverges if } p \leq 1 \end{array} \right.$$

Variations of these Series.

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n$$

$$\frac{1}{n}\left(\frac{1}{2}\right)^n \leq \underbrace{\left(\frac{1}{2}\right)^n}$$

The partial sums

$$S_N = \sum_{n=1}^{N} \frac{1}{n}\left(\frac{1}{2}\right)^n \leq \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n$$

S_N is increasing.

on the other hand

$$S_N \le \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n$$

$$S_N < 1$$

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n$$

converges.

Comparison test 1 If

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

(1) $0 \le a_n \le b_n$ for $n \ge N_0$

(2) $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges

Example $a_n = \frac{1}{n}\left(\frac{1}{2}\right)^n$

$b_n = \left(\frac{1}{2}\right)^n$

$0 \le a_n \le b_n$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges

$\sum_{n=1}^{\infty} a_n$ converges.

Example $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+7}$

$n^2 + 3n + 7 > n^2$

$$\boxed{\text{If } a > b > 0 \quad \frac{1}{a} < \frac{1}{b}}$$

$$\boxed{\begin{gathered} 3 > 2 \\ 1/3 < 1/2 \end{gathered}}$$

$$0 < \frac{1}{n^2+3n+7} < \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+7}$ Converges.

Comparison test 2

$$\sum_{n=1}^{\infty} a_n , \quad \sum_{n=1}^{\infty} b_n$$

$$0 < a_n \leq b_n$$

If $\sum_{n=1}^{\infty} a_n$ diverges

then $\sum_{n=1}^{\infty} b_n$ diverges.

Example: $$\sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

$$\frac{\ln n}{\text{[illegible]}} \geq \ln 2 \quad \text{for } n \geq 2.$$

$$\frac{\ln n}{n} \geq \frac{\ln 2}{n}$$

$$\sum_{n=1}^{\infty} \frac{\ln 2}{n} = \ln 2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3n}}$$

"looks like"

when n is large $\frac{1}{\sqrt{n^2+3n}} \sim \frac{1}{n}$

$$n^2+3n > n^2$$

$$\frac{1}{n^2+3n} < \frac{1}{n^2}$$

$$\underbrace{\frac{1}{\sqrt{n^2+3n}}} < \underbrace{\frac{1}{n}}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$\frac{1}{\sqrt{n^2+3n}} > \frac{C}{\sqrt{n^2}}$$

$$n^2+3\underline{\underline{n}} \leq n^2+3\underline{\underline{n}}^2 \leq 4n^2$$

$$n^2+3n \leq 4n^2$$

$$\frac{1}{n^2+3n} \geq \frac{1}{4n^2}$$

$$\frac{1}{\sqrt{n^2+3n}} > \frac{1}{2n}$$

But $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges and

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3n}}$.

Rephrase the comparison Test.

$$\sum_{n=1}^{\infty} a_n \quad , \quad \sum_{n=1}^{\infty} b_n$$

$$a_n > 0 \quad , \quad b_n > 0$$

$$0 < C_1 < \frac{a_n}{b_n} < C_2$$

Then Either

(1) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge

OR

(2) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4n^2+1}}$$

For n large

$$n^3+4n^2+1 \sim n^3$$

$$\sqrt{n^3+4n^2+1} \sim n^{3/2}$$

$$\frac{1}{\sqrt{n^3+4n^2+1}} \sim \frac{1}{n^{3/2}}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is candidate for comparison.

$$a_n = \frac{1}{\sqrt{n^3+4n^2+1}} \qquad b_n = \frac{1}{n^{3/2}}.$$

$$\frac{a_n}{b_n} = \frac{\frac{1}{\sqrt{n^3+4n^2+1}}}{1/n^{3/2}}$$

$$= \frac{n^{3/2}}{\sqrt{n^3+4n^2+1}} =$$

$$= \frac{n^{3/2}}{\sqrt{n^3\left(1+4/n+1/n^3\right)}} =$$

$$= \frac{\cancel{n^{3/2}}}{\cancel{n^{3/2}}\sqrt{1+4/n+1/n^3}} = \frac{1}{\sqrt{1+\frac{4}{n}+\frac{1}{n^3}}}$$

$$\frac{a_n}{b_n} = \frac{1}{\sqrt{1+\frac{4}{n}+\frac{1}{n^3}}}$$

$$c_1 < \frac{a_n}{b_n} \leq c_2$$

<u>Conclusion</u>: Either

Also Converges $\longrightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4n^2+1}}$ and

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}}$$ Converges