Lesson 23 The Comparison Tests – Continuation.

The Comparison test

(1) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

$$\boxed{0 \le a_n \le b_n}$$

If $\sum_{n=1}^{\infty} b_n$ converges then

$\sum_{n=1}^{\infty} a_n$ converges as well

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then

$\sum_{n=1}^{\infty} b_n$ also diverges.

$\underline{\text{If}}$ $a_n > 0$, $b_n > 0$

$$\boxed{0 < c_1 \le \frac{a_n}{b_n} \le c_2}$$

$$\underline{\underline{a_n}} \le c_2 b_n \leftarrow$$

$$\longrightarrow \underline{a_n \ge c_1 b_n}.$$

If $\sum_{n=1}^{\infty} a_n$ converges. then $\sum_{n=1}^{\infty} b_n$ also converges.

If $\sum_{n=1}^{\infty} b_n$ converses, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges,

$\sum_{n=1}^{\infty} b_n$ diverges.

If $\sum_{n=1}^{\infty} b_n$ diverges,

$\sum_{n=1}^{\infty} a_n$ diverges

Conclusion: If $a_n > 0, b_n > 0$

$$0 < C_1 \le \frac{a_n}{b_n} \le C_2$$

Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

Converge or both diverge.

If $a_n > 0$, $b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

$L \neq 0$, $L \neq \infty$

$\frac{a_n}{b_n}$ are close to L as $n \to \infty$

This implies that

$$0 < C_1 \le \frac{a_n}{b_n} \le C_2$$

The Limit Comparison Theorem.

If $a_n > 0$, $b_n > 0$

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0}$$

$L \neq 0$, $L \neq \infty$

Then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

converge or

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+8n^2}}}$$

Converges.

When n is large

$$a_n = \frac{1}{\sqrt{n^5+8n^2}} = \frac{1}{\sqrt{n^5(1+8/n^3)}}$$

$$a_n = \frac{1}{n^{5/2}\sqrt{1+8/n^3}}$$

$$\boxed{\text{Set } b_n = \frac{1}{n^{5/2}}}$$

Find a candidate for b_n.

$$\cancel{\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/n^{5/2}}{\quad}}$$

$$\frac{a_n}{b_n} = \frac{\dfrac{1}{n^{5/2}\sqrt{1+8/n^3}}}{1/n^{5/2}}$$

$$= \frac{1}{\sqrt{1+8/n^3}}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+8/n^3}} = 1$$

Either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

converge or diverge together.

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$. Converges

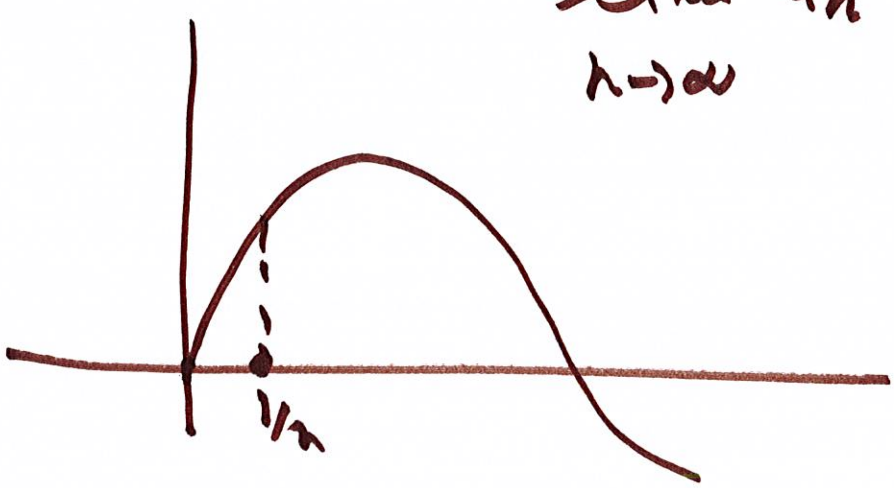
$$\sum_{n=1}^{\infty} \sin(1/n)$$

$$\sum_{n=1}^{\infty} \sin(1/n^2)$$

(1) $$\sum_{n=1}^{\infty} \sin(1/n).$$

$a_n = \sin(1/n) > 0.$

$\lim_{n\to\infty} a_n = 0$



Compare it to $\sum_{n=1}^{\infty} 1/n$

$b_n = 1/n$.

$$\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} =$$

(I) $1/n = x$ $\quad \lim_{x \to 0} \frac{\sin x}{x} = 1.$

(II) Think of n as a continuous variable

Apply L'Hopital's rule.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} =$$

$$= \lim_{n\to\infty} \frac{\cos(1/n)\cdot(-1/n^2)}{-1/n^2}$$

$$= \lim_{n\to\infty} \cos(1/n) = 1.$$

Since $\sum_{n=1}^{\infty} 1/n$ diverges

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

$\sum_{n=1}^{\infty} \sin(1/n)$ diverges.

$$\sum_{n=1}^{\infty} \sin\left(1/n^2\right)$$

Compare this to

$$\boxed{\sum_{n=1}^{\infty} 1/n^2 .}$$

$$\lim_{n \to \infty} \frac{\sin\left(1/n^2\right)}{1/n^2} \qquad 1/n^2 = x$$

$$\lim_{\substack{\cancel{n \to \infty} \\ x \to 0}} \frac{\sin x}{x} = 1.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges, $\sum_{n=1}^{\infty} \sin\left(1/n^2\right)$ converges.

For what values of p

does

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^p}\right)$$

converge?

Compare this to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty} \frac{\sin(1/n^p)}{1/n^p} = 1.$$

<u>Conclusion:</u> $\displaystyle\sum_{n=1}^{\infty} \sin(1/n^p)$ converges only if $p>1$.

$$\boxed{\sum_{n=1}^{\infty} \left(e^{1/n} - 1 \right) \text{ diverges.}}$$

$$a_n = e^{1/n} - 1.$$

$$\lim_{n \to \infty} a_n = 0$$

Next we want to compare this to another series $\sum_{n=1}^{\infty} b_n$

Let's try $b_n = 1/n$.

$$\lim_{n\to\infty} \frac{e^{1/n}-1}{1/n} =$$

$$\lim_{n\to\infty} \frac{e^{1/n}\cdot\left(-\cancel{1/n^2}\right)}{\left(-\cancel{1/n^2}\right)}$$

$$= \lim_{n\to\infty} e^{1/n} = 1.$$

$$\sum_{n=1}^{\infty} \left[-\cos\left(1/n\right) + 1\right]$$

Compare to $\sum_{n=1}^{\infty} \underline{\underline{b_n}}$

$b_n = 1/n$

$$\lim_{n\to\infty} \frac{\cos(1/n) - 1}{1/n} =$$

$$= \lim_{n\to\infty} \frac{-\sin(1/n)\left(-1/n^2\right)}{-1/n^2}$$

$$= \lim_{n\to\infty} -\sin(1/n) = 0.$$

Try $\sum 1/n^2$

$$\lim_{n \to \infty} \frac{-\cos(1/n) + 1}{1/n^2}$$

$$= \lim_{n \to \infty} \frac{+\sin(1/n)(-1/n^2)}{-2/n^3}$$

$$= \lim_{n \to \infty} \div \frac{\sin(1/n)}{-2/n} = +1/2.$$