# Lesson 23 Alternating Series

Average $\sim 60\%$

A - 90 - 100

B - 70 - 80

C - 60

$C^-$ - 50

D - 40

# Series

$$\sum_{n=1}^{\infty} a_n.$$

**Test 0**: If $\lim_{n\to\infty} a_n$

either does not exist or

$\lim_{n\to\infty} a_n = L \neq 0$

The series diverges.

However if $\lim_{n\to\infty} a_n = 0$

then we have to analyze the series a little closer.

# Geometric Series

(1) $\displaystyle\sum_{n=1}^{\infty} r^n$

Converges if $|r| < 1$

Diverges if $|r| \geq 1$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \leq 1$

(3) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ Converges if $p > 1$

diverges if $p \leq 1$.

# Comparison tests.

$$\sum_{n=1}^{\infty} a_n \ , \ \cancel{b} \sum_{n=1}^{\infty} b_n .$$

(1) If $0 \le a_n \le b_n$

$$\sum_{n=1}^{\infty} a_n \text{ diverges} , \quad \sum_{n=1}^{\infty} b_n \text{ diverges}$$

(2) If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges

The limit comparison: If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0 , \ L \neq \infty$$

Then both series converge or both diverge.

# Alternating Series.

<u>Example.</u> $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

We know that

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{\underline{\underline{n}}} \qquad b_n = 1/n$$

$$S_1 = 1$$

$$S_2 = 1 - 1/2$$

$$S_3 = 1 - 1/2 + 1/3$$

$$S_4 = 1 - 1/2 + 1/3 - 1/4$$

odd terms:

$0 < S_1 = \underline{\underline{1}}$

$0 < S_3 = 1 - \underbrace{\frac{1}{2} + \frac{1}{3}} < S_1$

$0 < S_5 = \underbrace{1 - \frac{1}{2} + \frac{1}{3}} \underbrace{- \frac{1}{4} + \frac{1}{5}} < S_3$

$0 < S_7 = \underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \underbrace{\frac{1}{5} - \frac{1}{6}} + \frac{1}{7}$

$< S_5$

Conclusion $S_N$ converges

when $N$ is odd.

Same argument works for

$N$ even. So $S_N$ converges

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges.

# Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n .$$

(1) $b_n > 0$

—— " —— " ——

When does an alternating series converges?

## Particular case

(2) If $b_n$ is decreasing.

$$b_1 > b_2 > b_3 > b_4 > \cdots$$

(3) $\lim\limits_{n \to \infty} b_n = 0$.

If (1), (2) and (3)

happen

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

Converges.

Examples :

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}.$$ Converges.

$$b_n = \frac{1}{\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left( \sqrt{n+1} - \sqrt{n} \right)$$

$b_n = \sqrt{n+1} - \sqrt{n}$.

(1) Is $b_n > 0$

(2) Is $b_n$ decreasing?

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$$

$x+1 > x$, $\sqrt{x+1} > \sqrt{x}$

$1/\sqrt{x+1} < \frac{1}{\sqrt{x}}$

Conclusion:

$f(x)$ is decreasing.

$f(n) = b_n = \sqrt{n+1} - \sqrt{n}$

is decreasing.

(3) $\lim\limits_{n \to \infty} \left( \sqrt{n+1} - \sqrt{n} \right) = 0$

$$\sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\boxed{\lim_{n \to \infty} \left( \sqrt{n+1} - \sqrt{n} \right) = 0}$$

<u>Conclusion:</u> $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left( \sqrt{n+1} - \sqrt{n} \right)$

Converges.

$$|S_N - S| \leq b_{N+1}$$

Example:

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$S_{1000} = \sum_{n=1}^{1000} (-1)^{n-1} \frac{1}{n} \qquad b_n = \frac{1}{n}$$

$N = 1000$

$$\boxed{|S - S_{1000}| \leq \frac{1}{1001}}$$

Estimating the Sum of an alternate series.

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

(1) $b_n > 0$, (2) $b_n$ is decreasing

(3) $\lim_{n \to \infty} b_n = 0$

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n .$$

Can we estimate $|S - S_N|$

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!}$$

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n!}$$

<u>Question:</u> Find the smallest $N$ such that

$$|S - S_N| < 10^{-5}.$$

$$|S - S_N| \leq b_{N+1}$$

In this case $b_n = 1/n!$

$$|S - S_N| \leq \frac{1}{(N+1)!}$$

$$\frac{1}{(N+1)!} < 10^{-5}$$

$$(N+1)! > 10^5.$$

$\underline{N=1}$ $(N+1)! = 2! = 2$

$\underline{N=5}$: $(N+1)! = 6! = 6 \times 5 \times 3 \times 2$

$\underline{N=9}$

$$(N+1)! = 10! = \underbrace{10 \times 9} \times 8 \times 7 \times 6 \times 5$$

$$\times 3 \times 2 \times 1 =$$

$N = 10$

$$(N+1)! = (11)!$$

$$= 11 \times 10 \times 9 \times \cdots \times 1 =$$