# Lesson 24 Absolute and Conditional convergence; The ratio and root tests.

**Example:** $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Converges

However $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$a_n = (-1)^{n-1} \frac{1}{n}$, $|a_n| = \frac{1}{n}$

$\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$a_n = (-1)^{n-1} \frac{1}{n^2} \quad , \quad |a_n| = \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} |a_n|$ converges and $\sum_{n=1}^{\infty} a_n$ also converges.

If $\sum_{n=1}^{\infty} |a_n|$ converges we say that $\sum_{n=1}^{\infty} a_n$ converges

<u>ABSOLUTELY</u>

When $\sum_{n=1}^{\infty} a_n$ converges but

$\sum_{n=1}^{\infty} |a_n|$ does not converge

$\sum_{n=1}^{\infty} a_n$ converges conditionally.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$ converges conditionally

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$ converges absolutely.

Claim: If $\sum_{n=1}^{\infty} |a_n|$

converges, $\sum_{n=1}^{\infty} a_n$

also converges.

Why is that?

$$0 \le |a_n| + a_n \le 2|a_n|$$

The comparison test says

that If $\sum_{n=1}^{\infty} |a_n|$

converges, $\sum_{n=1}^{\infty} (|a_n| + a_n)$

converges.

$$\sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

$$= \sum_{n=1}^{\infty} (a_n + \cancel{|a_n|} - \cancel{|a_n|})$$

$$= \sum_{n=1}^{\infty} a_n \quad \text{Converges}$$

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Tests For Absolute Convergence.

# The Ratio test

$$\sum_{n=1}^{\infty} a_n \quad , \quad |a_n| > 0$$

$$\left| \frac{a_{n+1}}{a_n} \right| < r < 1.$$

for $n > N$

$$|a_{n+1}| < r |a_n|$$

$$|a_{n+2}| < r |a_{n+1}| < r^2 |a_n|$$

$$|a_{n+3}| < r^3 |a_n|$$

Conclusion

$$|a_{n+k}| < r^k |\underline{\underline{a_n}}|$$

$k = 1, 2, \ldots \qquad n > N$

$$\underbrace{\sum_{k=1}^{\infty} |a_{n+k}|} < |a_n| \underbrace{\sum_{k=1}^{\infty} r^k}$$

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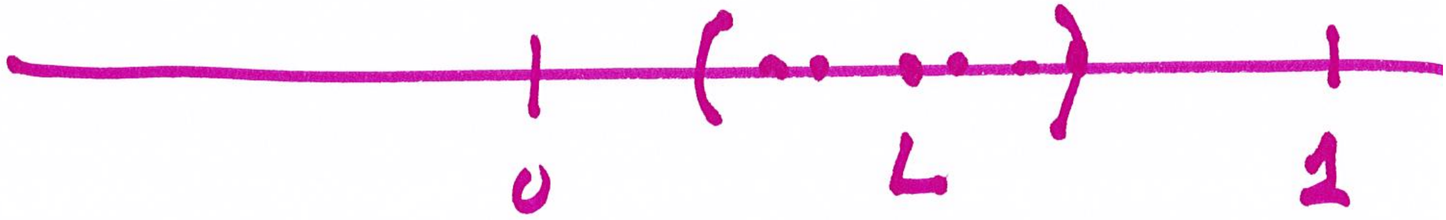
If $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

$\underline{L < 1}$




for $n > N$ $\quad \left|\dfrac{a_{n+1}}{a_n}\right| < r < 1$

## Ratio test Part I:

If $\displaystyle \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

and $L<1$

then $\displaystyle \sum_{n=1}^{\infty} |a_n|$ converges.

# Part II

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$

then $\sum_{n=1}^{\infty} |a_n|$ diverges.

In fact $\sum_{n=1}^{\infty} a_n$ diverges

If $\underline{L = 1}$ The test is useless.

Find the values of

$x$ such that

the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Converges.

Ex: $x = 1$ $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$x = -1$ $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$a_n = \frac{1}{n}, \quad \frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{1/n} = \frac{n}{n+1}$$

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n}{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

$$a_n = 1/n^2, \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

$$a_n = \frac{x^n}{n^2}, \quad x \neq 0$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right|$$

$$\underbrace{|x|} \cdot \frac{n^2}{(n+1)^2}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \cdot \lim_{n \to \infty} \frac{n^2}{(n+1)^2}$$

$$= |x|.$$

<u>Conclusion:</u> $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|.$

When $|x| < 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{|x^n|}{n^2}$ converse

If $|x|>1$.

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} \text{ diverges.}$$

$|x|=1$. , $\underline{\underline{x=1}}$ or $x=-1$

$\underline{x=1}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

The series converges if

$$|x| \le 1$$

diverges if $|x|>1$.

# The Root test

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

If $L < 1$, $\sum_{n=1}^{\infty} |a_n|$ converges

if $L > 1$ $\sum_{n=1}^{\infty} a_n$ diverges

if $L = 1$, the test is inconclusive.

$$\sum_{n=1}^{\infty} \left( \sin 1/n \right)^n$$

$$a_n = \left( \sin \frac{1}{n} \right)^n$$

$$a_n^{1/n} = \sin 1/n$$

$$\lim_{n \to \infty} a_n^{1/n} = 0 < 1$$

Converges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}-\frac{1}{4^n}\right)^n$$

$$a_n=\left(\frac{1}{2}-\frac{1}{4^n}\right)^n$$

$$|a_n|^{1/n}=\frac{1}{2}-\frac{1}{4^n}$$

$$\lim_{n\to\infty}|a_n|^{1/n}=1/2<1$$

$$\sum_{n=1}^{\infty}\left(1/2-1/4^n\right)^n$$ converges.

# Examples

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

$$a_n = \frac{10^n}{n!} \qquad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n}$$

$$= \frac{10^{n+1}}{10^n} \cdot \frac{n!}{(n+1)!}$$

$$= 10 \cdot \frac{1\,\cancel{n!}}{(n+1)\,\cancel{n!}} = \frac{10}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$$

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)!} \quad \text{converges.}$$