Lesson 25

The Root test and Review of Series

The Root test

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

If $L < 1$, $\sum_{n=1}^{\infty} |a_n|$ converges.

and so $\sum_{n=1}^{\infty} a_n$ converges

If $L > 1$ $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, the test is inconclusive.

<u>Example</u>. (1) $\sum_{n=1}^{\infty} \left(1+1/n\right)^{n^2}$

(2) $\sum_{n=1}^{\infty} \left(1/n\right)^{n}$.

(2) $\sum_{n=1}^{\infty} \left(1/n\right)^{n}$ $a_n = \left(1/n\right)^{n}$

$|a_n|^{1/n} = 1/n$ $\lim_{n\to\infty} |a_n|^{1/n} = 0$.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{n} \text{ converges.}$$

$$(1) \quad \sum_{n=1}^{\infty} \underbrace{\left(1+\frac{a}{n}\right)^{n^2}}_{a_n}$$

For what values of a this series converges.

$$a_n = \left(1 + a/n\right)^{n^2}$$

$$|a_n|^{1/n} = \left[\left(1+\frac{a}{n}\right)^{n^2}\right]^{1/n}$$

$$= \left(1+\frac{a}{n}\right)^{n}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = L}$$

$$\ln L = \lim_{n\to\infty} \ln\left(1+\frac{a}{n}\right)^n$$

$$= \lim_{n\to\infty} n \ln\left(1+\frac{a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+a/n\right)}{1/n}.$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+a/n}\cdot -a/n^2}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{a}{1+a/n} = a.$$

Conclusion:

$$L = \lim_{n\to\infty} \left(1 + a/n\right)^n$$

$$\ln L = a$$

$$L = e^a$$

$$\boxed{\lim_{n\to\infty} \left(1 + a/n\right)^n = e^a.}$$

The original question was:

$$\sum_{n=1}^{\infty} \left(1 + a/n\right)^{n^2}$$

$$a_n = \left(1 + \frac{a}{n}\right)^{n^2}$$

$$\lim_{n \to \infty} a_n^{1/n} = e^a.$$

When is $e^a > 1$, $a > 0$

The Series diverges if $a > 0$

$e^a < 1$, $a < 0$

The series converges if $a < 0$.

<u>when $a = 0$</u>. $e^a = 1$.

the series is $\sum_{n=1}^{\infty} 1$ diverges.

Convergence Tests

Test for divergence:

$$\sum_{n=1}^{\infty} a_n$$

Test if $\lim_{n\to\infty} |a_n| = 0$

If not, the series diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+5}{n+200}$$

$$a_n = (-1)^n \frac{n+5}{n+200}$$

$$|a_n| = \frac{n+5}{n+200} \qquad \lim_{n\to\infty} |a_n| = 1.$$

$$\sum_{n=1}^{\infty} (-1)^n \cos(1/n^{10})$$ diverges

$$\sum_{n=1}^{\infty} (-1)^n e^{1/n^4}.$$ diverges.

$$a_n = (-1)^n \cos\left(\frac{1}{n^{10}}\right)$$

$$|a_n| = \cos\left(\frac{1}{n^{10}}\right), \lim_{n\to\infty} |a_n| = 1$$

If $\sum_{n=1}^{\infty} a_n$, $\lim_{n \to \infty} a_n = 0$.

Then we have to use the tests we learned.

<u>The geometric Series</u>

$$\sum_{n=1}^{\infty} r^n$$

$a_n = r^n$, $|a_n| = |r|^n$

$$|a_n|^{1/n} = |r|$$

If $|r| < 1$, the series converges

if $|r| \geq 1$ diverges.

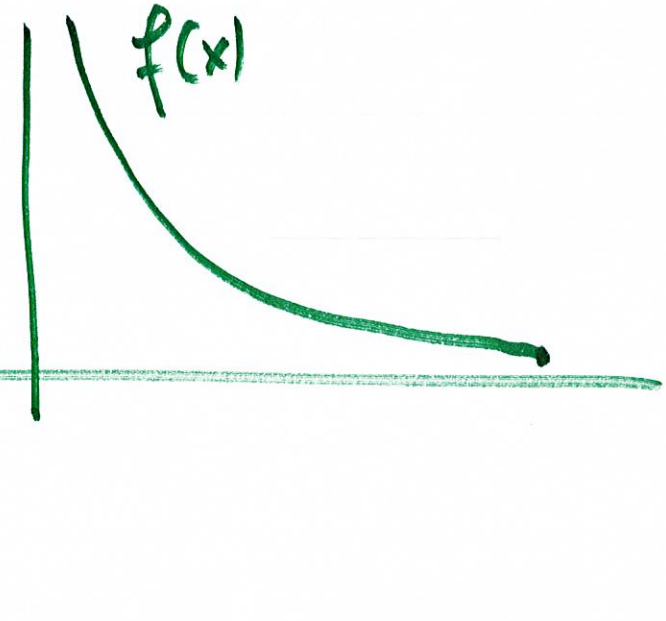
$$\sum_{n=1}^{\infty} (-1)^n \text{ diverges.}$$

$$a_n = (-1)^n, \quad |a_n| = 1.$$

$$\boxed{\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \text{if } |r| < 1}$$

The integral test

If $f(x)$



(1) $f(x) > 0$ if $x > 1$

(2) f is decreasing

(3) $\lim_{x \to \infty} f(x) = 0$

$$\sum_{n=1}^{\infty} f(n)$$ converges if and only if

$$\int_1^{\infty} f(x)\,dx$$

converges.

<u>Example</u>: $$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$ converges.

$$f(x) = \frac{1}{x^2+1}$$

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \arctan x \Big|_1^{\infty}$$

$$= \pi/2 - \pi/4 = \pi/4.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ converges if $p > 1$
diverges if $p \leq 1$

(1) $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$ Converges if $p > 1$
diverges if $p \leq 1$.

(2) $$f(x) = \frac{1}{x(\ln x)^p}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$$ $u = \ln x$

$du = \frac{dx}{x}$

$$= \int_{\ln 2}^{\infty} \frac{du}{u^p}$$ converges if $p > 1$
diverges if $p \leq 1$

Alternating Series

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

(1) $b_n > 0$

(2) $b_n > b_{n+1} > b_{n+2} > \cdots$

(3) $\lim_{n \to \infty} b_n = 0$

The series converges <u>and</u>

$$S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$$

$$|S - S_N| \le b_{N+1}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$$

Find the smallest number of terms of the series such that the corresponding partial sum approximates the series with an error $< 10^{-3}$.

$$|S - S_N| \leq b_{N+1} \qquad b_n = \frac{1}{n^3}$$

$$|S - S_N| \leq \frac{1}{(N+1)^3} < \frac{1}{10^3}$$

$$(N+1)^3 > 10^3$$

$$N+1 > 10$$

$$\underbrace{N > 9}.$$

$$S_{10} = \sum_{n=1}^{10} (-1)^n \frac{1}{n^3}.$$

The Comparison tests

(1) $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$

$$0 < a_n < b_n$$