Lesson 26 Power Series

Example: $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

provided $|x| < 1$.

This series is a sum of powers of $\underline{\underline{x}}$

and is called a "power series".

In general:

$$\sum_{n=0}^{\infty} c_n x^n$$ a power series.

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$ = a power series centered at a.

We want to study the convergence of these series.

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

always converges for $\underline{x=a}$

We adopt the convention

$$0^0 = 1.$$

$\underline{n=0}$: $c_0 (0)^0 = c_0$.

When $\underline{x \neq a}$ $c_0 (x-a)^0 = c_0$

Using this convention:

$$c_0 (x-a)^0 = c_0$$

<u>Example:</u> Find all the values of x for which the series

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \qquad 0! = 1$$

converge.

$\underline{x \neq 0}$ The terms of the series

$$a_n = \frac{2^n x^n}{n!}.$$

Try the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$a_{n+1} = 2^{n+1} \cdot \frac{x^{n+1}}{(n+1)!}$$

$$a_n = 2^n \frac{x^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\boxed{2^{n+1}}\,|x^{n+1}|}{(n+1)!} \cdot \frac{n!}{\boxed{2^n}\,|x^n|}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2 \cdot |x|}{n+1}$$

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0}$$

Conclusion: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

Converges for any x.

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$$

For what values of x does this converge?

Again we use the ~~the~~ ratio test

$$a_n = 2^n \frac{x^n}{n^2}.$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$a_{n+1} = 2^{n+1} \frac{x^{n+1}}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n x^n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = 2|x| \frac{n^2}{(n+1)^2}$$

$$\lim_{n \to \infty} 2|x| \frac{n^2}{(n+1)^2} = \underline{\underline{2|x|}}.$$

<u>Conclusion:</u> The series converges absolutely if $\boxed{2|x| < 1}$

diverges if $2|x| > 1$.

How about $|x| = \frac{1}{2}$?

$$|x| = \frac{1}{2} \begin{cases} x = \frac{1}{2} \\ x = -\frac{1}{2}. \end{cases}$$

when $\underline{x = \frac{1}{2}}$:

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2} = \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}.$$

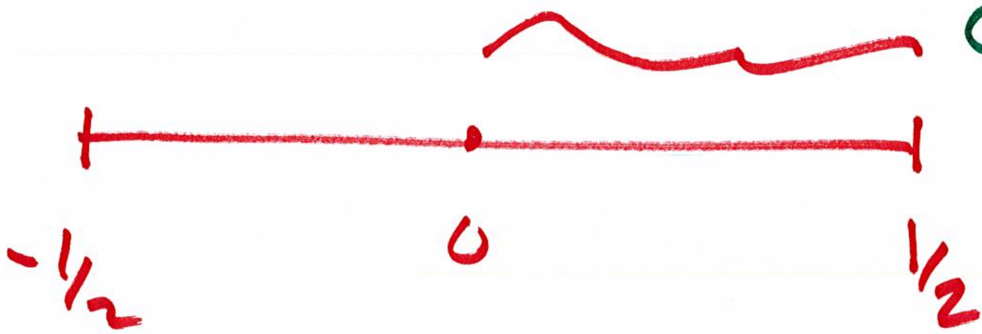
when $x = \frac{1}{2}$ $$= \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$x = -\frac{1}{2}$ $$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Conclusion: The Series $\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$ converges for all x in $[-\frac{1}{2}, \frac{1}{2}]$.

The interval $[-\frac{1}{2}, \frac{1}{2}]$ is called the interval of convergence of the series.

The radius of convergence $= \frac{1}{2}$.



Find the radius and the interval of convergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} (x-1)^n$$

This series is ~~cented~~ centered at 1.

Step 1: $a_n = (-1)^n \frac{3^n}{n} (x-1)^n$

Step 2: Apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = ?$$

$$|a_{n+1}| = \frac{3^{n+1}}{n+1} \cdot |x-1|^{n+1}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}}{n+1} \cdot |x-1|^{n+1} \times \frac{n}{3^n |x-1|^n}$$

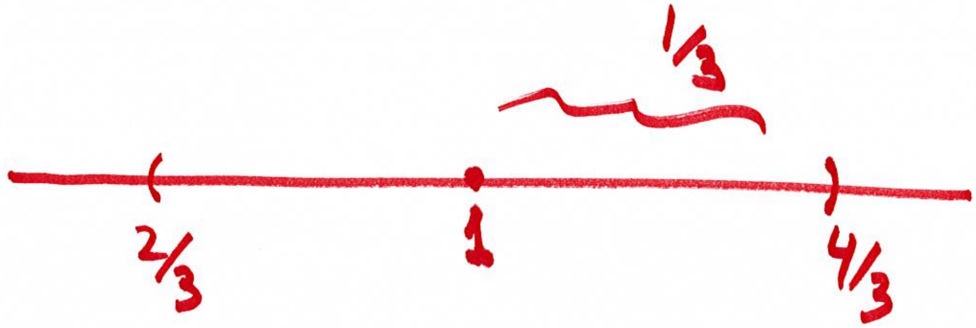
$$\left| \frac{a_{n+1}}{a_n} \right| = \underbrace{3\,|x-1|} \cdot \underbrace{\frac{n}{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 3|x-1|$$

If $3|x-1| < 1$, the series converges absolutely

If $3|x-1| > 1$, the series diverges.

$3|x-1| < 1$; $\boxed{|x-1| < \frac{1}{3}}$



The radius of convergence is $\frac{1}{3}$.

Test the end points of the interval:

$x = \frac{2}{3}$, $x = \frac{4}{3}$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} \underbrace{(x-1)}_{}{}^{n}$$

$x = 4/3$; $x - 1 = 1/3$

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} \cdot \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges.

$x = 2/3$ $\quad$ $x - 1 = -1/3$

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} \left(-\frac{1}{3}\right)^n =$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cancel{3^n}}{n} \frac{(-1)^n}{\cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges.

Conclusion: The

Series

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} (x-1)^n$$

Converges if x is in

$$\left(\frac{2}{3}, \frac{4}{3}\right]$$

Includes $4/3$.

Excludes $2/3$

$$\sum_{n=1}^{\infty} n^n (x-5)^n.$$

$\underline{x=5}$

$$a_n = n^n (x-5)^n$$

$\underline{x \neq 5}$ $|a_n|^{1/n} = n\,|x-5|$ $\qquad |x-5| > 0$

$$\lim_{n\to\infty} n\,|x-5| = \infty$$

The series $\displaystyle\sum_{n=1}^{\infty} n^n (x-5)^n$

only converges when $x=5$.