Lesson 28

Representation of Functions By Power Series.

Starting Point

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{provided } |x|<1.$$

We interpret this formula

$\frac{1}{1-x}$ can be represented as

a power series centered at 0, provided $|x|<1$.

Can we represent

$$\frac{1}{1-x}$$ by a

power series ~~centered~~
centered at 3.

$$\frac{1}{1-x} \overset{?}{=} \sum_{n=0}^{\infty} c_n (x-3)^n.$$

$$\frac{1}{1-x} = \frac{1}{1-(x-3+3)}$$

$$= \frac{1}{-2-(x-3)} = -\frac{1}{2+(x-3)}$$

$$\frac{1}{1-x} = \frac{-1}{2+(x-3)}$$

$$\boxed{\frac{1}{1-y} = \sum_{n=0}^{\infty} \underbrace{y^n}, \quad |y|<1}$$

$$-\frac{1}{2+(x-3)} = -\frac{1}{2}\,\frac{1}{1+\left(\frac{x-3}{2}\right)}$$

$$= \underbrace{-\frac{1}{2}}\,\frac{1}{1-\underbrace{\left(-\frac{x-3}{2}\right)}_{y.}}.$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(- \frac{x-3}{2} \right)^n$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-1}{2} \right)^n (x-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} (x-3)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} (x-3)^n$$

Provided $\left| \frac{x-3}{2} \right| < 1.$

$$\frac{1}{1+x} = \frac{1}{1-\underbrace{(-x)}}$$

$$= \sum_{n=0}^{\infty} (-x)^n \qquad |x| < 1$$

$$\boxed{\frac{1}{1+\underline{x}} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n \qquad |x| < 1$$

$$\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1}.$$

If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$|x-a| < R$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

$$+ \; C$$

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad |x|<1$$

$$\underbrace{\int \frac{dx}{1+x^2}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\int \frac{dx}{1+x^2} = \arctan x$$

$|x|<1$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

<u>Set $x=0$</u> $\qquad 0 = 0 + C.$

$$\boxed{C=0}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}, \quad |x|<1$$

$$\underbrace{\int \frac{dx}{1+x}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$|x|<1$

Set $x=0$ $\quad 1+x>0$

$C=0$

$$\ln 1 = 0 = 0 + C$$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

provided $|x| < 1$.

This is a power series representation of $\arctan x$.

$n+1 = m$

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$$

provided $\underline{|x| < 1.}$

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}.$$

$x = 1/2$

$$\boxed{\ln\left(3/2\right) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{2^m m}}$$

Set $\boxed{x=1}$ (This can be done by Abel's theorem)

$$\boxed{\ln 2 = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

provided $|x| < 1$.

$$\boxed{\underline{x=1} \quad \pi/4 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}$$

$$\frac{1}{x^2+4x+5} = \sum_{n=1}^{\infty} c_n (x+2)^n$$

Start from $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

or $\displaystyle \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

$$\frac{1}{x^2+4x+5} = \frac{1}{(x+2)^2+1}$$

$$\frac{1}{1+\underbrace{(x+2)^2}} = \sum_{n=0}^{\infty} (-1)^n (x+2)^{2n}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a| < R$

What is the meaning of

c_n ?

We want to express

c_n in terms of f.

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$c_3(x-a)^3 + \cdots$$

$$\boxed{f(a) = c_0}$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2$$

$$+ \cdots$$

Set $x = a$

$$f'(a) = c_1.$$

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots$$

$$f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2}.$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$