Lesson 29 Section 11.10

Taylor and Maclaurin Series

Last Lesson Representation of Functions by Power Series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1.$$

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}, \quad |x|<1$$

$$\int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^n \,dx + C$$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$|x| < 1.$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$|x| < 1.$

$m = n+1 \quad , \quad n = m-1$

$$\boxed{\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}}$$

$|x| < 1$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n , \quad |x|<1$$

replace x by x^2

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} , \quad |x|<1$$

$$\underbrace{\int \frac{dx}{1+x^2}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad |x|<1}$$

$$f(x) = \sum_{n=0}^{\infty} \underline{\underline{c_n}} \, (x-a)^{\underline{\underline{n}}}$$

$$|x-a| < R$$

Ex:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \, x^{\underline{2n+1}}$$

$\underline{a=0}$ $\underline{\text{Even terms}}$ $c_0 = c_2 = c_4 = \dots = 0$

$c_1 = 1$ $c_5 = \frac{1}{5}$

$n=1$ $c_3 = \frac{-1}{3}$ $c_7 = -\frac{1}{7} \dots$

Relation between c_n

and $f(x)$?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n =$$

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3$$

$$+ \dots$$

Set $\underline{x=a}$: $f(a) = c_0$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3 (x-a)^2$$

$$+ 4c_4 (x-a)^3 + \dots$$

Set $\underline{x=a}$: $\boxed{f'(a) = c_1.}$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$c_0 = f(a); \quad c_1 = f'(a)$$

$$c_2 = \frac{f''(a)}{2}, \quad c_3 = \frac{f^{(3)}(a)}{3!}$$

$$c_4 = \frac{f^{(4)}(a)}{4!}, \ldots$$

If $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a| < R$.

We have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for $|x-a| < R$.

The series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

is called the Taylor series of f centered at a.

When $a=0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is the Maclaurin Series of f.

Example. $f(x) = \frac{1}{x}$, $x \neq 0$

Find the taylor series of $f(x)$ centered at 3.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n.$$

$$f(x) = \frac{1}{3 + x - 3}$$

$$= \frac{1}{3} \frac{1}{1 + \boxed{\frac{x-3}{3}}} \leftarrow y$$

Remember that $\sum_{n=0}^{\infty} (-1)^n y^n = \frac{1}{1+y}$

if $|y| < 1$

$y = \frac{x-3}{3}$

$$f(x) = \frac{1}{x} = \frac{1}{3} \frac{1}{1 + \frac{x-3}{3}}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-3}{3}\right)^n \qquad \left|\frac{x-3}{3}\right| < 1$$

The series is

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n$$

$$\left|\frac{x-3}{3}\right| < 1.$$

Find the Maclaurin Series of

$$f(x) = e^x.$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

$$f(x) = e^x; \quad f(0) = 1.$$

$f'(x) = e^x, \quad f'(0) = 1$

$f^{(2)}(x) = e^x, \quad f^{(2)}(0) = 1$

$f^{(n)}(x) = e^x, \; f^{(n)}(0) = 1$

The Maclaurin Series of

e^x is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

<u>Question</u> Is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

for every x.

Compute the Maclaurin Series of

$\sin x$ and $\cos x$.

$f(x) = \sin x$

Its Maclaurin Series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Compute $f^{(n)}(0)$ for $n = 0, 1, 2, \ldots$

$$f(x) = \sin x, \quad f(0) = 0$$

$$f'(x) = \cos x, \quad f'(0) = 1$$

$$f^{(2)}(x) = -\sin x, \quad f^{(2)}(0) = 0$$

$$f^{(3)}(x) = -\cos x, \quad f^{(3)}(0) = -1.$$

$$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0.$$

$$f^{(2n)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

$$f^{(2n+1)}(0) = (-1)^n$$

The Maclaurin Series of

$\sin x$ is

$$(*) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Find the Maclaurin Series

of

$\sin 2x$.

Replace x by $2x$ in

the formula (*)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1}.$$