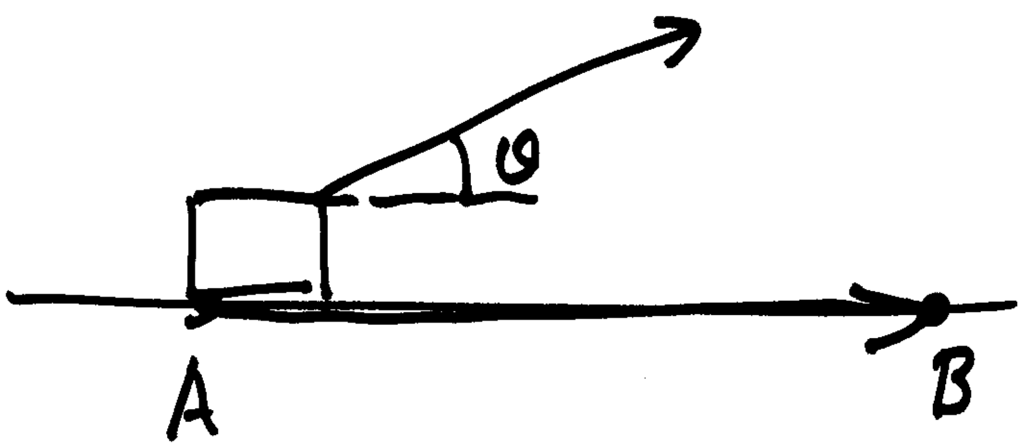
Lesson 3 Section 12.3

The Dot Product

Example:

$\vec{F}$ = force

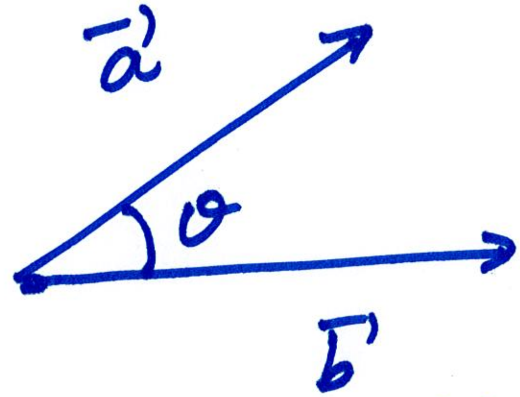


Work $|\vec{F}| \cdot |\overrightarrow{AB}| \cdot \cos\theta$

= The dot product of $\vec{F}$ and $\overrightarrow{AB}$

$\vec{a}$, $\vec{b}$ two vectors

<u>Define</u> $\vec{a} \cdot \vec{b} = |\vec{a}|.|\vec{b}| \cos\theta.$

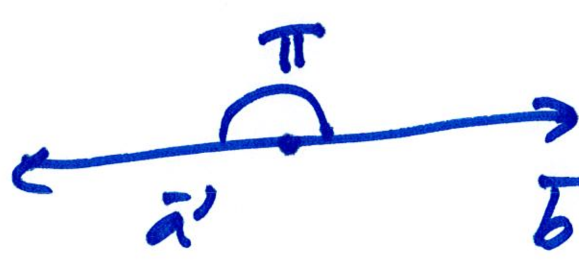


θ = angle between $\vec{a}$ and $\vec{b}$

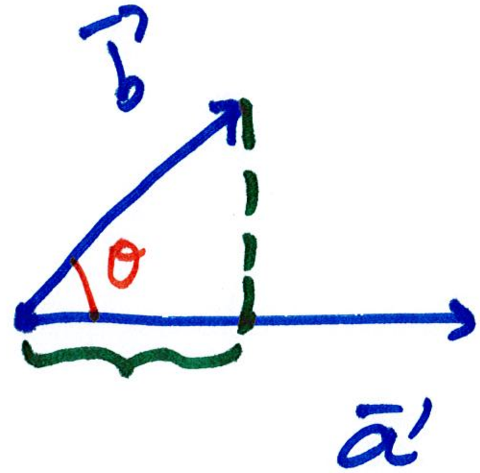
$0 \leq \theta \leq \pi.$

$\theta = 0$

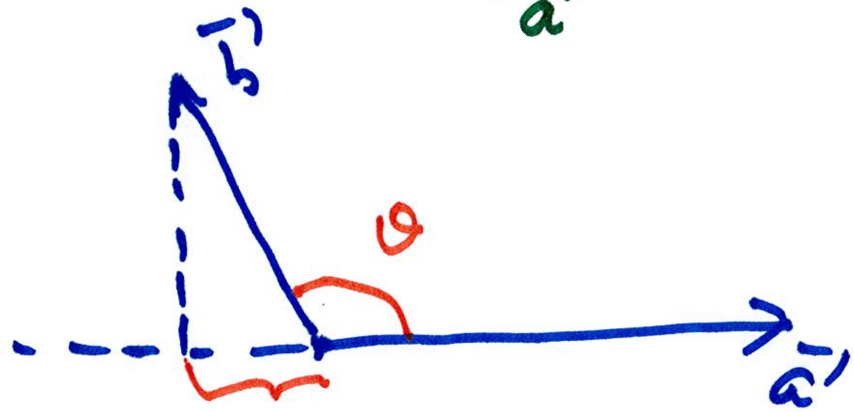
$\theta = \pi$

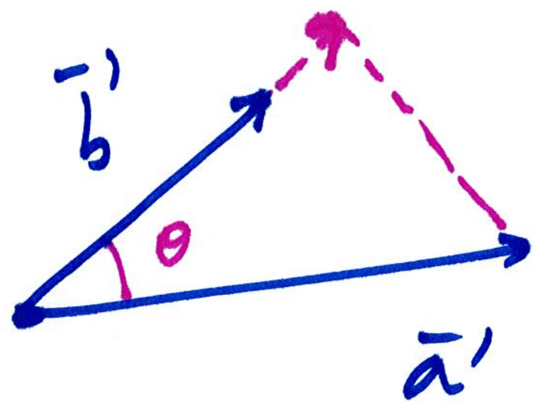


Vector Projections



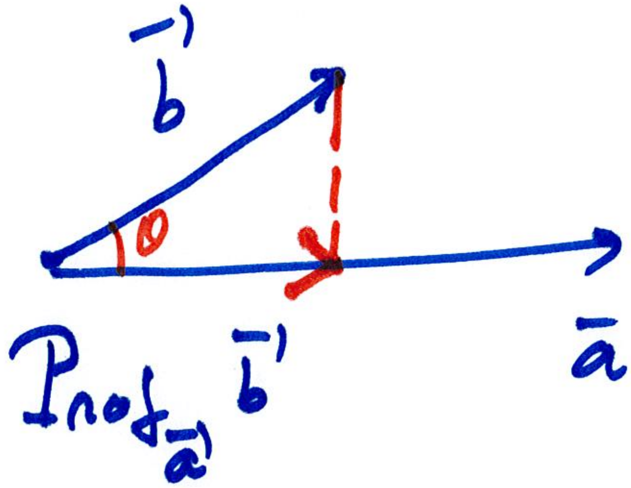
Component: $\text{Comp}_{\vec{a}} \vec{b} = |\vec{b}| . \cos\theta .$





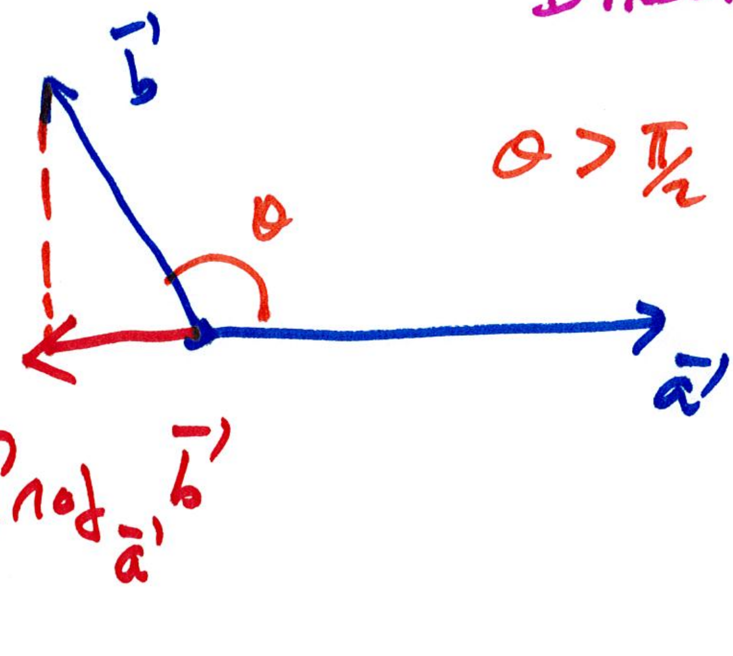
$$\mathrm{Comp}_{b}\,\vec{a} = |\vec{a}|.\cos\theta.$$

Projection:



$$\text{Proj}_{\vec{a}}\vec{b} = |\vec{b}|.\cos\theta.\underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{Direction of }\vec{a}}.$$

$\theta > \frac{\pi}{2}$, $\cos\theta \leq 0$.

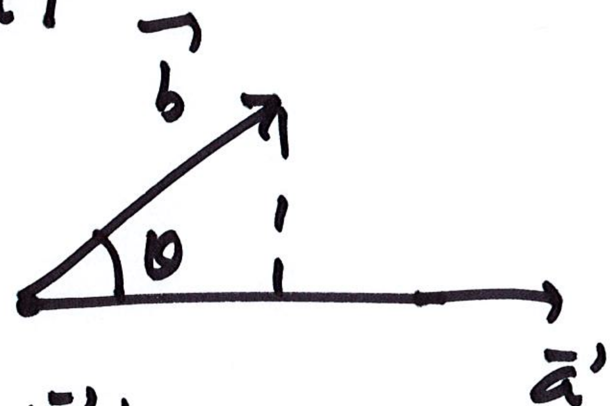


We want to represent these quantities using

$\vec{a} \cdot \vec{b}$.

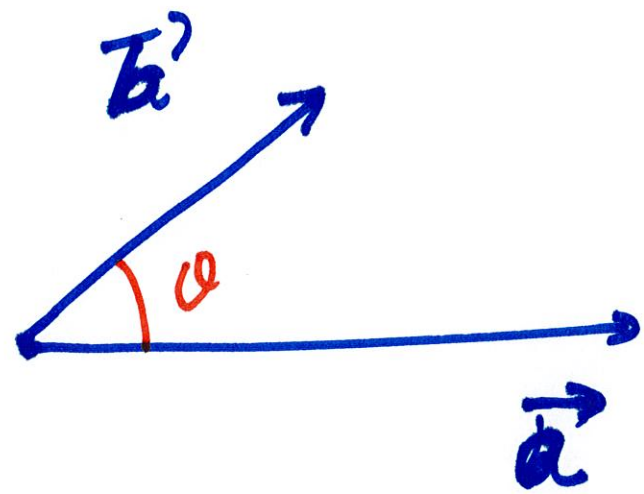
$$\text{Comp}_{a} \vec{b} = \underbrace{|\vec{b}| \cdot \cos\theta} = \frac{\overbrace{|\vec{a}| \cdot |\vec{b}| \cdot \cos\theta}^{\vec{a} \cdot \vec{b}}}{|\vec{a}|}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}.$$



$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

$$\boxed{\text{Comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}}$$



$$\text{Proj}_{\vec{a}}\,\vec{b} = \underbrace{|\vec{b}|.\cos\theta}\ \frac{\vec{a}}{|\vec{a}|}$$

$$\text{Proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$$

$$\boxed{\text{Proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\ \vec{a}}$$

Use components to compute Dot products, angles, etc.

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

<u>Example:</u> $\vec{a} = \langle 1, 0, 2 \rangle$

$\vec{b} = \langle -1, 2, 1 \rangle$

$$\vec{a} \cdot \vec{b} = -1 + 0 + 2 = 1.$$

Examples

1) Find the angle between

$\vec{a} = \langle 1, 2, 1 \rangle$ and $\vec{b} = \langle -1, 0, 2 \rangle$

On one hand

$$\vec{a} \cdot \vec{b} = -1 + 0 + 2 = 1.$$

on the other hand

$$\vec{a} \cdot \vec{b} = |\vec{a}| . |\vec{b}| . \cos\alpha.$$

$$|\vec{a}| = \sqrt{6}, \quad |\vec{b}| = \sqrt{5}.$$

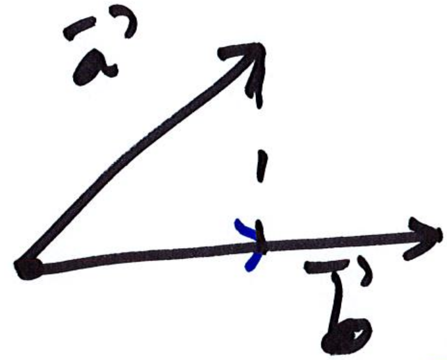
$$\vec{a} \cdot \vec{b} = \sqrt{5} . \sqrt{6} . \cos\alpha = \sqrt{30}\cos\alpha.$$

$$\vec{a}\cdot\vec{b} = 1 = \sqrt{30}\cos\alpha$$

$$\boxed{\cos\theta = 1/\sqrt{30}}$$

Projections:

$$\mathrm{Proj}_{\vec{b}}\,\vec{a} = \,?$$



$$\mathrm{Proj}_{\vec{b}}\,\vec{a} = |\vec{a}|\cdot\cos\theta\cdot\frac{\vec{b}}{|\vec{b}|}.$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b} = \frac{1}{5}\cdot\langle -1, 0, 2\rangle$$

$\vec{a} = \langle 1, 0, 2 \rangle$

$\vec{b} = \langle x, 1, 3 \rangle$

Find x such that $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\vec{a} \cdot \vec{b} = |\vec{a}| . |\vec{b}| . \cos\theta = 0$$

$$\cos\theta = 0 \quad \theta = \pi/2 .$$

$$\vec{a}, \vec{b} = x + 6 = 0 \quad \boxed{x = -6}$$

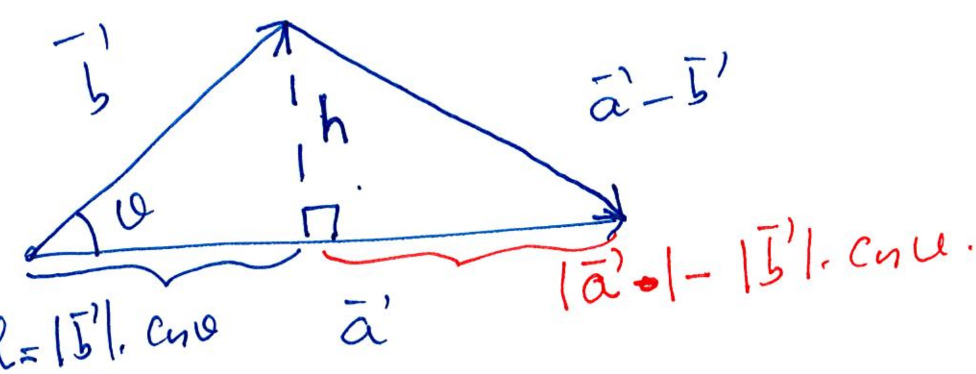
Extra note about dot products

Claim: If $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = a_1 b_1 + a_2 b_2.$$

The three dimensional case is similar.

Step 1 Use the law of cosines.



$h = |\vec{b}| \cdot \sin\theta.$

Using the Pythagorean theorem we have.

$$|\vec{a} - \vec{b}| = h^2 + (|\vec{a}| - |\vec{b}| \cos\theta)^2.$$

$$= |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 + |\vec{b}|^2 \cos^2\theta - 2|\vec{a}| \cdot |\vec{b}| \cos\theta$$

$$= |\vec{b}|^2 (\cos^2\theta + \sin^2\theta) + |\vec{a}|^2 - 2|\vec{a}| \cdot |\vec{b}| \cos\theta.$$

$$\boxed{|\vec{a} - \vec{b}|^2 = |\vec{b}|^2 + |\vec{a}|^2 - 2|\vec{a}|.|\vec{b}|.\cos\theta}$$

Equation A

Step 2: We write this in coordinates.

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$|\vec{a} - \vec{b}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$= a_1^2 + b_1^2 - 2a_1 b_1 + a_2^2 + b_2^2 - 2a_2 b_2.$$

$$= (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1 b_1 + a_2 b_2).$$

$$= |\vec{a}|^2 + |\vec{b}|^2 - 2(a_1 b_1 + a_2 b_2).$$

$$\boxed{|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(a_1 b_1 + a_2 b_2)}$$

Equation B

From Equations A and B we get

$$|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|.|\vec{b}|\cos\theta = |\vec{a}|^2 + |\vec{b}|^2 - 2(a_1 b_1 + a_2 b_2)$$

So we conclude: $\boxed{|\vec{a}|.|\vec{b}|\cos\theta = a_1 b_1 + a_2 b_2}$