# Lesson 30 Taylor and Maclaurin Series – Continuation

We start with a function

$f(x)$

The Taylor series of $f(x)$

centered at $a$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Question 1: Does the series converge for $|x-a|<R$?

Question 2: Is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for $|x-a|<R$?

## Examples:

$$f(x) = e^x$$

$$f^{(n)}(0) = 1.$$

The Maclaurin series of

$e^x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

**Question 1:** Does $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Converge in $|x-0| < R$?

$$a_n = \frac{x^n}{n!} \qquad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = x \cdot \frac{n!}{(n+1)!}$$

$$n! = n(n-1)\cdot(n-2)\ldots 1$$

$$(n+1)! = (n+1)\, \overbrace{n\,(n-1)\ldots 1}$$

$$= (n+1)\, n!$$

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

$$\frac{a_{n+1}}{a_n} = \frac{x}{n+1}. \qquad \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

$$< 1$$

for any $x$.

Conclusion : $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Converges for any $x$.

The radius of convergence is equal to infinity.

Question #2

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x - e^0 = \int_0^x \frac{d}{ds} e^s \, ds$$

$$\underbrace{e^x - 1} = \int_0^x e^s \, ds$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

$$e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots$$

$$- \frac{x^n}{n!} = R_n(x)$$

$$\boxed{\lim_{n \to \infty} R_n(x) = 0}$$

$e^x$ can be approximated by a polynomial of degree $n$

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\underbrace{e^x - 1} = \int_0^x \underbrace{e^s}\, ds$$

$$1 = T_0(x)$$

$$e^s = \int_0^s e^t\, dt + 1$$

$$e^s - 1 = \int_0^s e^t\, dt$$

$$e^x - 1 = \int_0^x \left[ 1 + \int_0^s e^t\, dt \right] ds$$

$$e^x - 1 = x + \left( \int_0^x \left( \int_0^s e^t\, dt \right) ds \right)$$

$$\underbrace{e^x - 1 - x} = \int_0^x \left( \int_0^s e^t\, dt \right) ds$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

In some cases we know this is true

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

↑ This is the Maclaurin Series of $\frac{1}{1-x}$.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad |x|<1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

$|x|<1.$

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Find the Maclaurin Series

of

$e^{3x}$.

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} 3^n \frac{x^n}{n!}$$

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$$4\underbrace{e^x} + e^{3x} = 4\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$+ \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \frac{4}{n!} + \frac{3^n}{n!} \right) x^n.$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Find the power series expansion of $\sin 4x$

$$\sin 4x = \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1}}{(2n+1)!} x^{2n+1}.$$

Find the Maclaurin

series of

$$\int \left( \frac{\sin x - x}{x^3} \right) dx.$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\frac{\sin x - x}{x^3} = \frac{1}{x^3} \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\frac{\sin x - x}{x^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-2}$$

$$\int \frac{\sin x - x}{x^3} \, dx =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n-1}}{(2n-1)} + C$$

Approximate $\sin 1$.

With an error less than

$10^{-4}$.

$\Longrightarrow$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}$$

This is an alternating series

$$\sum_{n=0}^{\infty} (-1)^n \, b_n \qquad b_n = \frac{1}{(2n+1)!}$$

$$\left| \sin 1 - \sum_{n=0}^{3} (-1)^n \frac{1}{(2n+1)!} \right| < 10^{-4}!$$

$$\left| \sin 1 - \sum_{n=0}^{N} (-1)^n \overbrace{\frac{1}{(2n+1)!}}^{b_n} \right|$$

$$\leq b_{N+1}$$

$$b_n = \frac{1}{(2n+1)!}, \quad b_{n+1} = \frac{1}{(2n+3)!}$$

We Want

$$\frac{1}{(2N+3)!} < 10^{-4}$$

$$(2N+3)! > 10^4.$$

$$2N+3 = 9 \qquad 2N = 6$$

$$\boxed{N = 3}$$