Lesson 32 Calculus With Parametric Curves.

Exam 3: Average 60

Median 64.

A - 82

B - 73

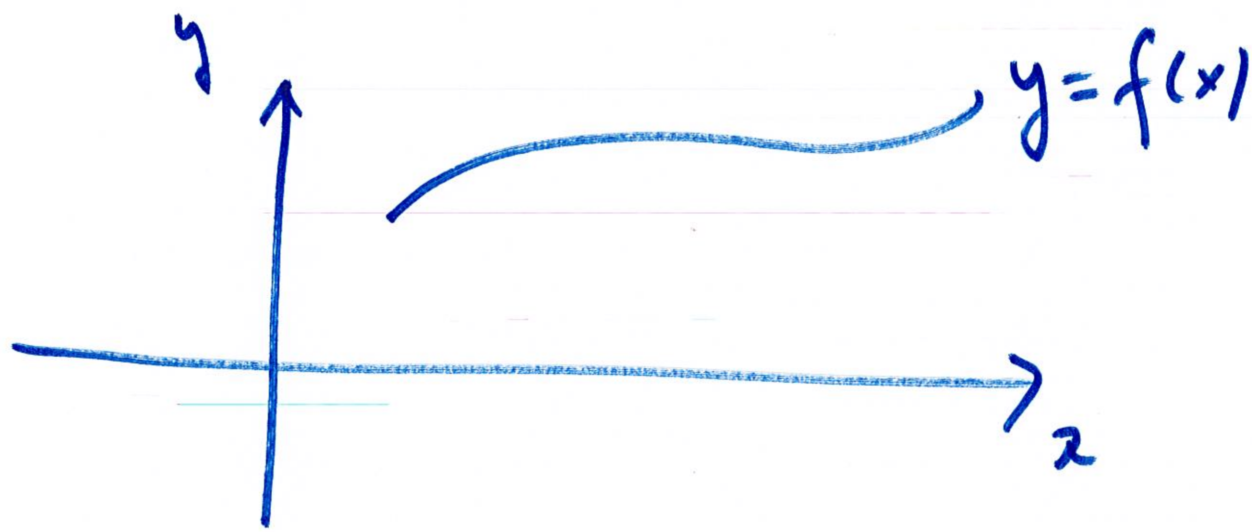
C - 55 and 64

D^+ to C^- — 46.

$$y = y(t)$$

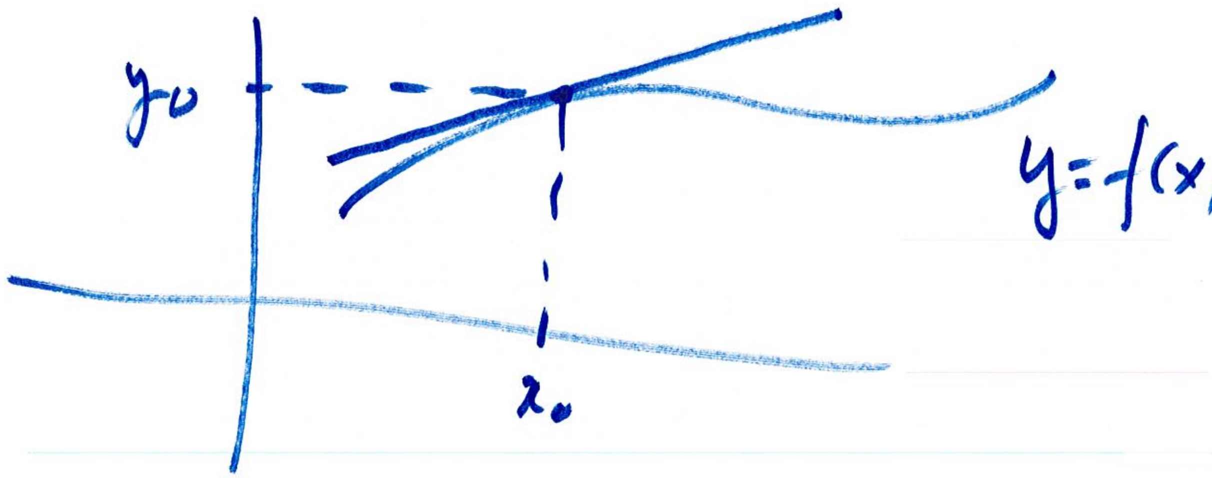
$$x = x(t)$$

<u>Example</u>: $y = f(x)$



$x = t, \quad y = f(t).$

$y = f(x)$

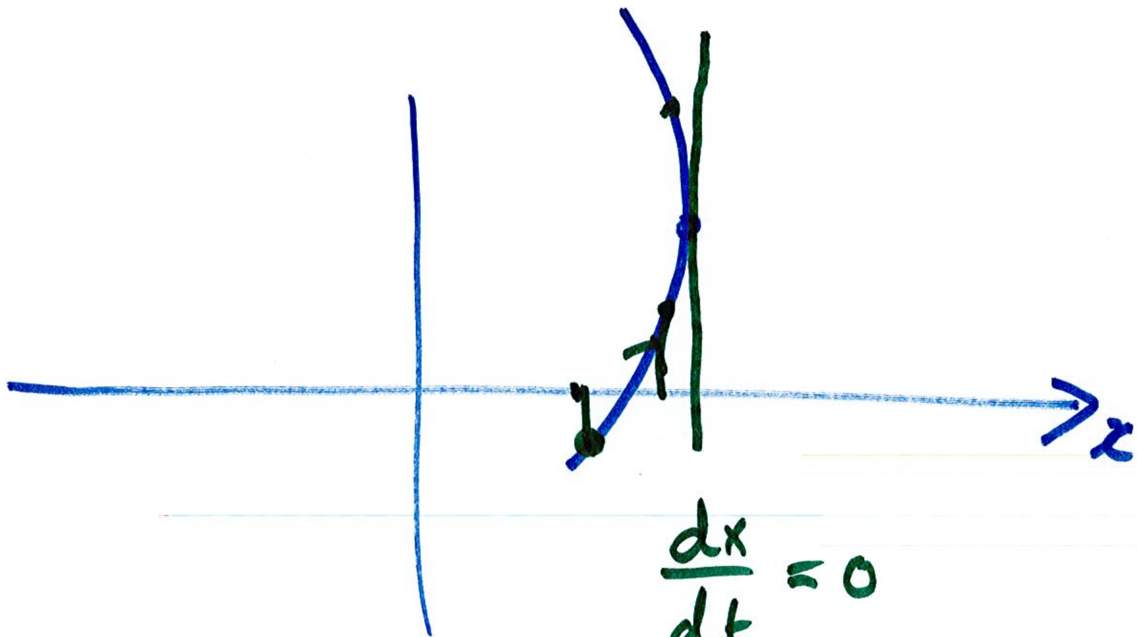


Slope of the tangent line

$\text{Slope} = \dfrac{dy}{dx} = f'(x)$

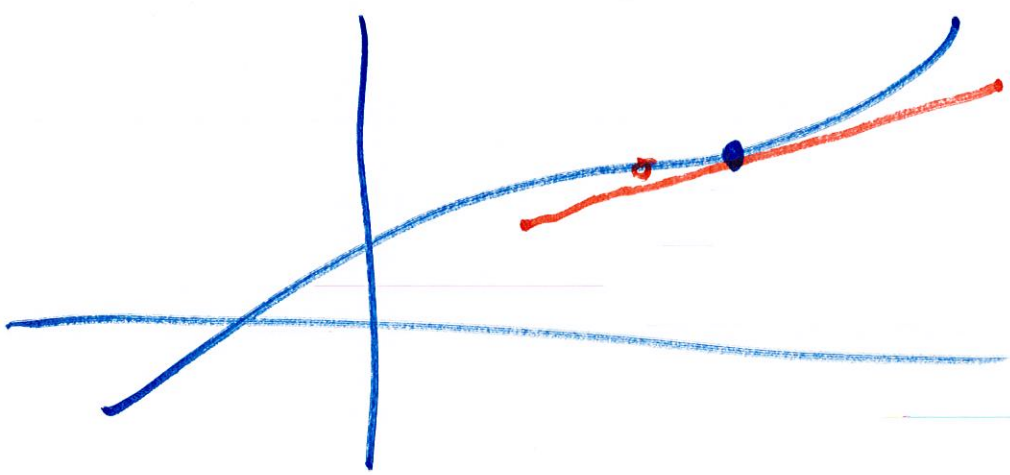
<u>Question:</u> Given $y = y(t)$
$x = x(t)$

find $\dfrac{dy}{dx}$, if possible.



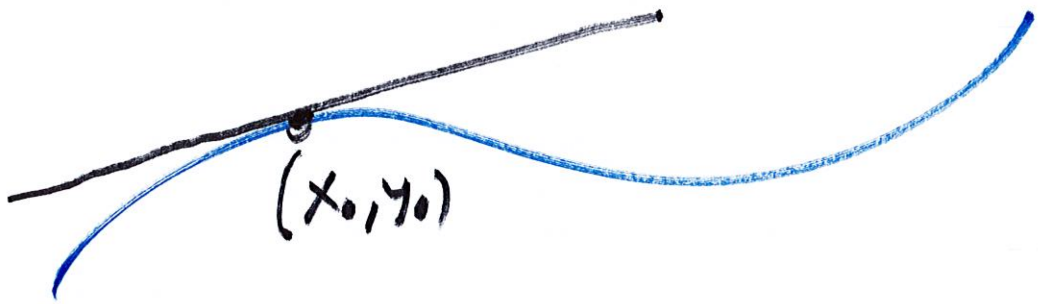
$\frac{dx}{dt} = 0$

$\frac{dx}{dt} \neq 0$, tangent line is not vertical



Example: $y = t^2 + 4t + 1$

$x = t^3 + 8t + 2.$



Find $\frac{dy}{dx}$ at $(x_0, y_0) = (11, 6)$

We have $y = y(t)$

$x = x(t)$

If $\frac{dx}{dt} \neq 0$ one can write

$t = t(x)$

$$y = y(t) \quad , \quad \text{but } t = t(x)$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}}$$

but $$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

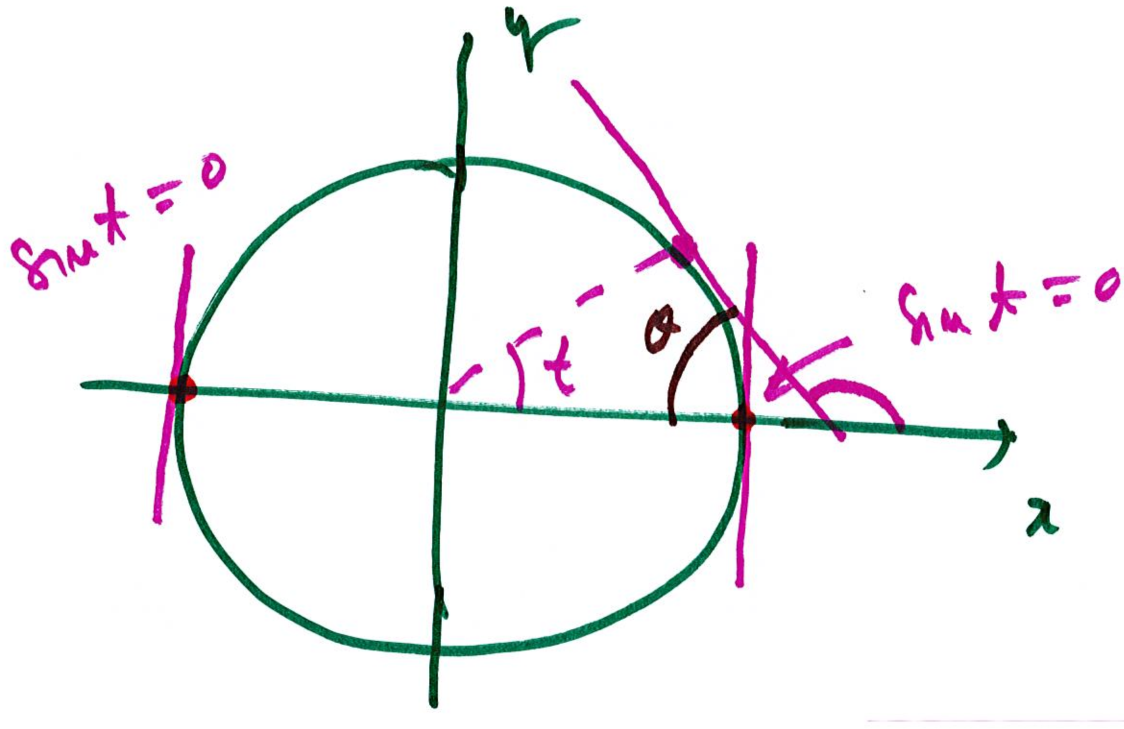
$$y = t^2 + 4t + 1$$

$$x = t^3 + 8t + 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+4}{3t^2+8}$$

when $\underline{t = 1}$: $y = 6$, $x = 11$.

$$\underline{\boxed{\frac{dy}{dx} = \frac{6}{11}}}$$



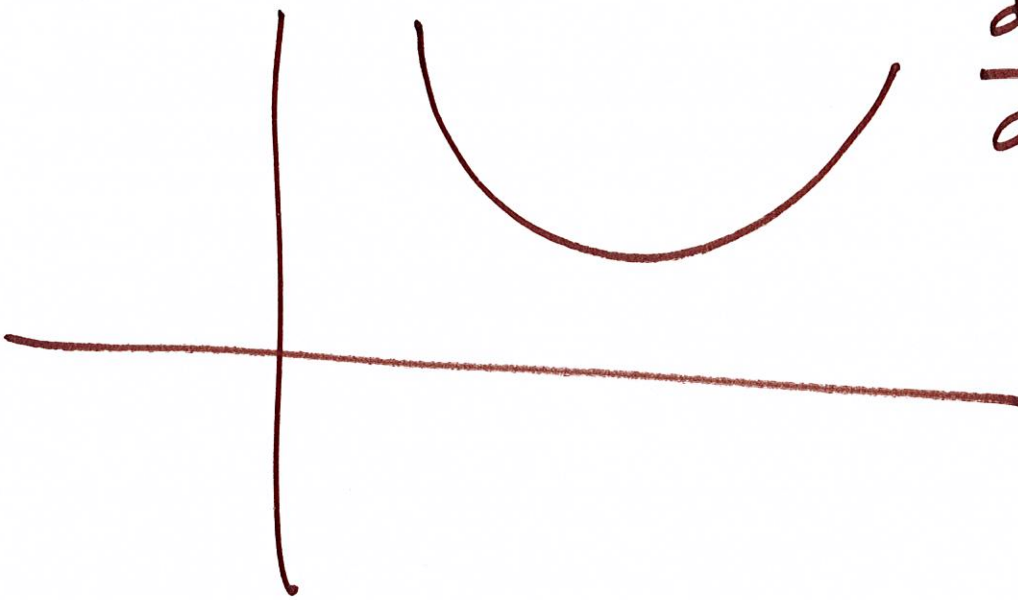
$x = \cos t, \quad y = \sin t.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

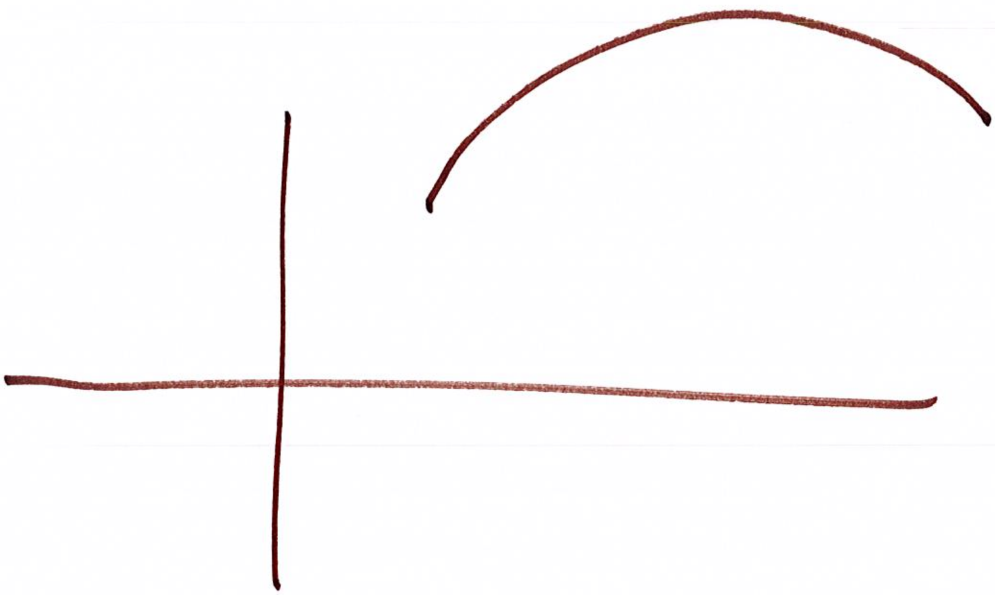
$$\boxed{\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\frac{\cos t}{\sin t}}$$

$\sin t \neq 0$

Concavity



$$\frac{d^2y}{dx^2} > 0$$



$$\frac{d^2y}{dx^2} < 0$$

Given $y = y(t)$, $x = x(t)$

Find $\frac{d^2y}{dx^2}$.

$$(*)\quad \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.}$$

y is arbitrary

Find $\frac{d^2y}{dx^2}$. $\qquad \frac{dx}{dt} \neq 0$

Use $(*)$ for $\frac{dy}{dx}$ in place of y.

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Example:

$$y = 3t^2 + 4t$$

$$x = t^4 + 8t^2 + 3t.$$

Find $\dfrac{d^2y}{dx^2}$.

Step 1: Find $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\boxed{\frac{dy}{dx} = \frac{6t+4}{4t^3+16t+3}}$$

Step 2 :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

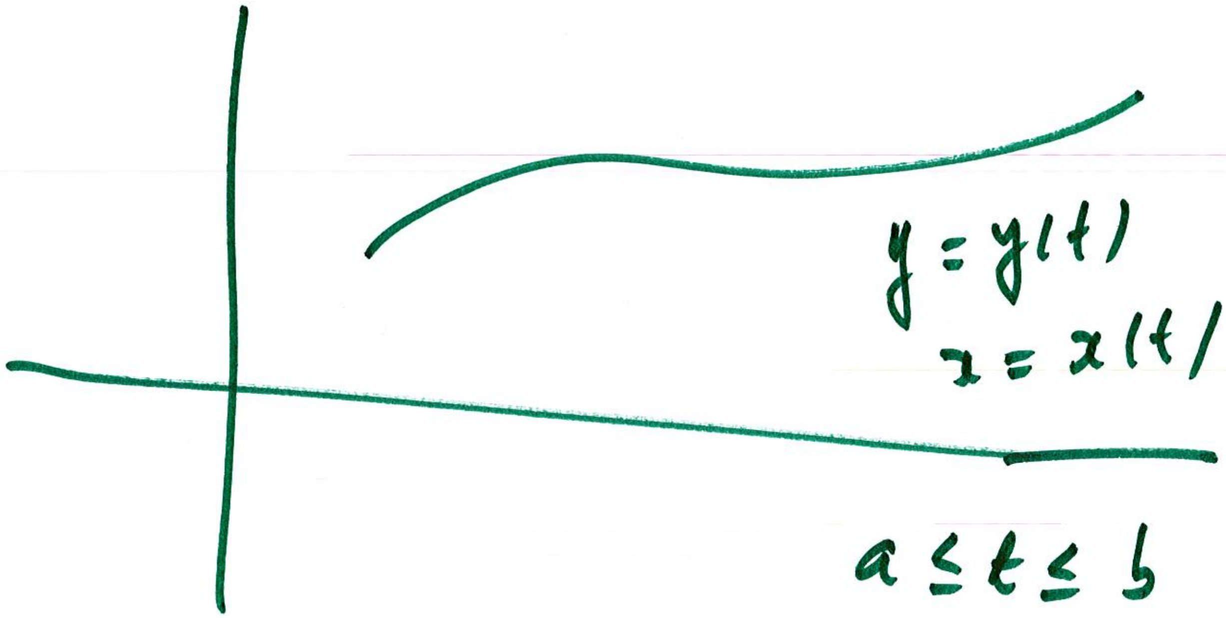
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{6t+4}{4t^3+16t+3}\right)}{4t^3+16t+3}$$

$$= \frac{\frac{6(4t^3+16t+3)-(6t+4)(12t^2+16)}{(4t^3+16t+3)^2}}{4t^3+16t+3}$$

$$= \frac{24t^3+96t+18-72t^3-96t-48t^2-64}{(4t^3+16t+3)^3}$$

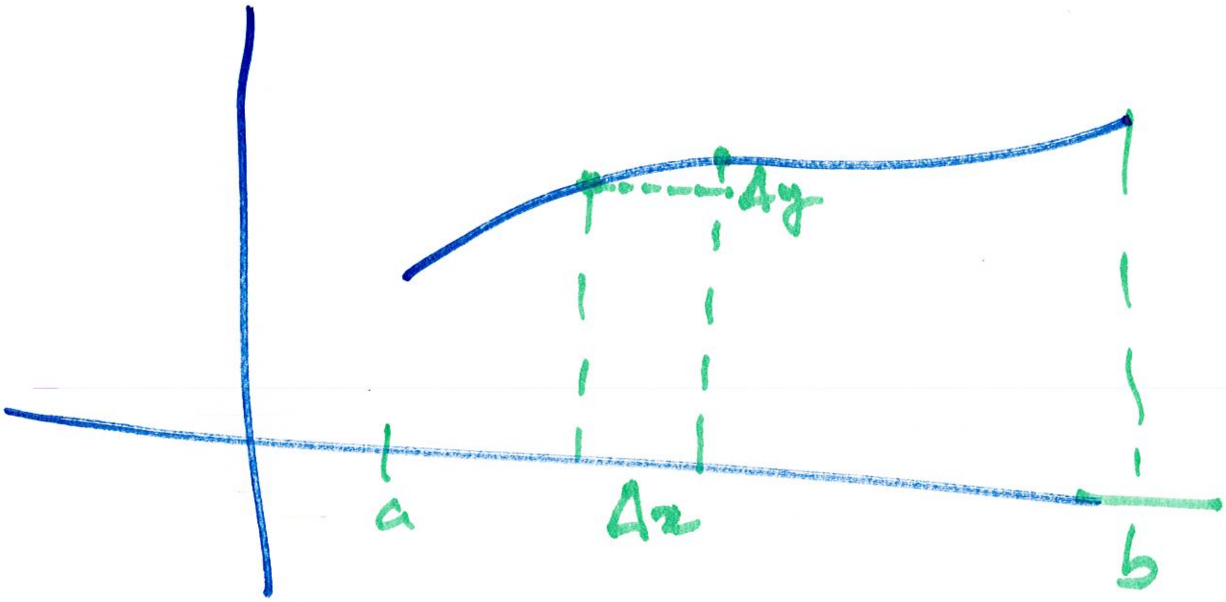
Arc Length



We have studied the case

$$y = f(x) \quad a \le x \le b$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$



$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$L = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t.$$

$$\sim \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$$

In the case of a graph

$$x = t, \quad y = f(t)$$

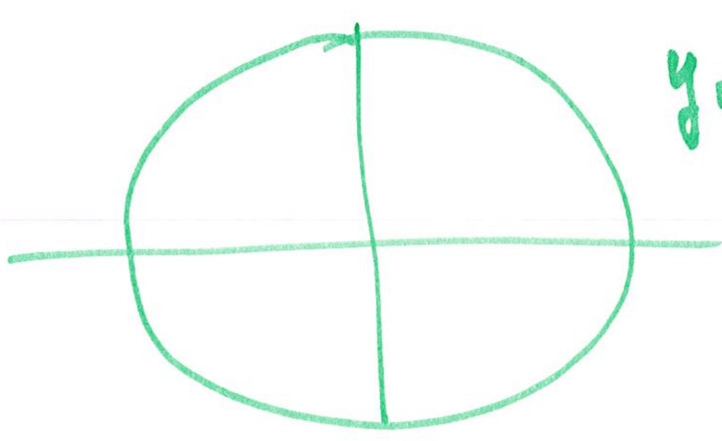
$$L = \int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

<u>Example:</u> Find the length of a circumference of radius R.

$x = R\cos t$

$y = R\sin t$

$0 \le t \le 2\pi$



$$\frac{dx}{dt} = -R\sin t$$

$$\frac{dy}{dt} = R\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = R^2\cos^2 t + R^2\sin^2 t$$

$$= R^2$$

$$L = \int_0^{2\pi} \sqrt{R^2}\, dt = 2\pi R.$$