Lesson 35 Last one.

Appendix H

Complex Numbers

Solve $x^2 + 1 = 0$

$$x^2 = -1.$$

Introduce $i = \sqrt{-1}$.

$$\boxed{i^2 = -1.}$$

$x^2 = -1$ has two solutions $x = i$ and $x = -i$

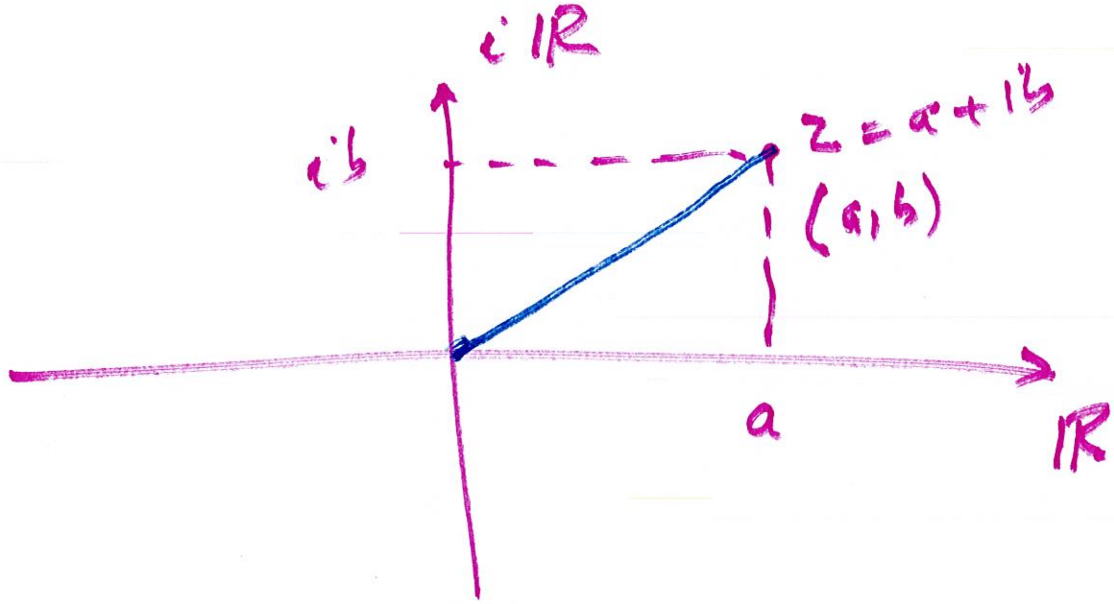
A Complex number

Imaginary part of z.

$z = a + ib$, a, b real numbers.

Real Part

We identify complex numbers to points on the plane



Arithmetic: $z = a + ib$, $w = c + id$

$z + w = (a + c) + i(b + d)$

$z - w = (a - b) + i(c - d)$

Multiplication:

$$z \cdot w = (a+ib)(c+id)$$

$$= ac + iad + ibc + i^2 bd$$

but $i^2 = -1$

$$z \cdot w = (ac - bd) + i(ad+bc).$$

Ex: $(1+i)(3-i)$

$$= 3 - i + 3i - i^2 = 4 + 2i.$$

$$\boxed{z = a + ib}$$

Absolute value: $|z| := |\langle a,b\rangle|$

$$= \sqrt{a^2+b^2}$$

when $b=0$: $z = a$, $|z| = \sqrt{a^2}$

$|z| = |a|.$

Quotient: $z = a+ib$, $w = c+id$.

$\frac{z}{w}$, $w \neq 0$ either $c \neq 0$ or $d \neq 0$

Complex Conjugate

$$z = a+ib$$

$$\bar{z} = \overline{a+ib} = a-ib$$

Ex: $\overline{1+2i} = 1-2i$

$$z \cdot \bar{z} = (a+ib)(a-ib)$$

$$= a^2 - \cancel{iab} + \cancel{iab} - i^2 b$$

$$= a^2 + b^2 \qquad \boxed{z \cdot \bar{z} = |z|^2}$$

$$\frac{z}{w} = \frac{a+ib}{c+id}$$

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z \cdot \bar{w}}{|w|^2}.$$

$$\boxed{\frac{z}{w} = \frac{z \cdot \bar{w}}{|w|^2}}$$

<u>Ex:</u> $\frac{1+3i}{2+5i} = \frac{(1+3i)\cdot(2-5i)}{4+25}$

$$= \frac{1}{29}\left(2 - 5i + 6i - 15i^2\right)$$

$$= \frac{1}{29}\left(17 + i\right).$$

Polar Representation.

Define e^z where $z = a+ib$

$$e^z = e^{a+ib} = e^a \cdot e^{ib}$$

Define e^{ib}. Here we use

The Maclaurin Series of e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substitute $x = ib$.

$$e^{ib} = \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} = A + iB$$

$$e^{ib} = \sum_{n=0}^{\infty} i^n \frac{b^n}{n!}$$

when n is even $n = 2k$

$$i^n = (i)^{2k} = (i^2)^k = (-1)^k$$

when n is odd $n = 2k+1$

$$i^n = (i)^{2k+1} = i \cdot i^{2k}$$

$$= i(-1)^k.$$

$$e^{ib} = \sum_{n=0}^{\infty} (i)^n \frac{b^n}{n!}$$

$$= \sum_{n \text{ even}} (i)^n \frac{b^n}{n!} + \sum_{n \text{ odd}} (i)^n \frac{b^n}{n!}$$

$\boxed{n = 2k}$ $\qquad$ $n = 2k+1$

$$= \sum_{k=0}^{\infty} (i)^{2k} \frac{b^{2k}}{(2k)!} + \sum_{k=0}^{\infty} (i)^{2k+1} \frac{b^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i(-1)^k \frac{b^{2k+1}}{(2k+1)!}$$

$$e^{ib} = \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!}} + i \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!}}$$

$$\boxed{e^{ib} = \cos b + i \sin b}$$

$b = \pi$

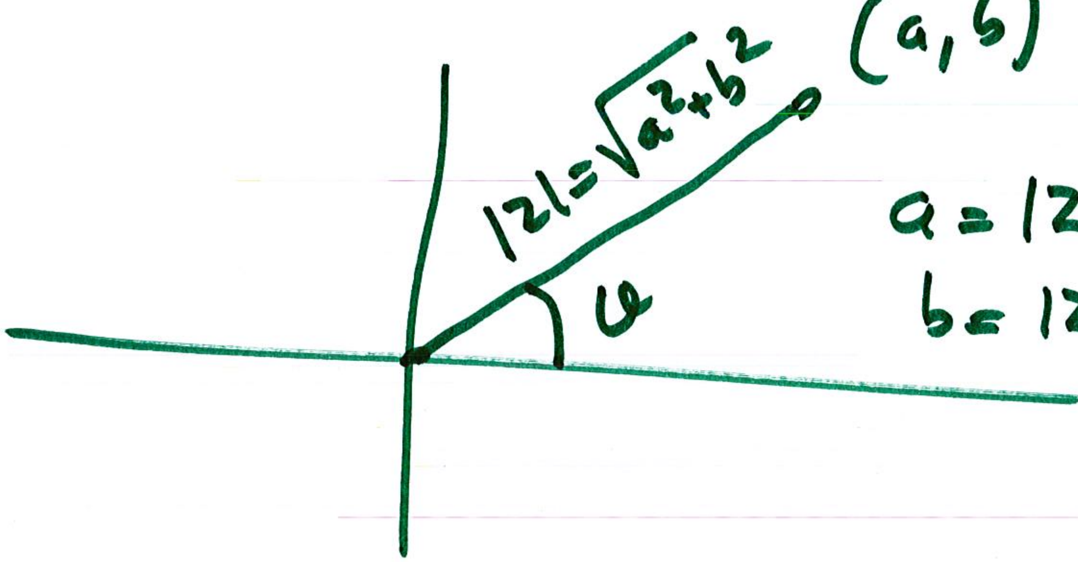
$$e^{i\pi} = \cos \pi$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Polar Formula for Complex numbers

$$z = a + ib$$



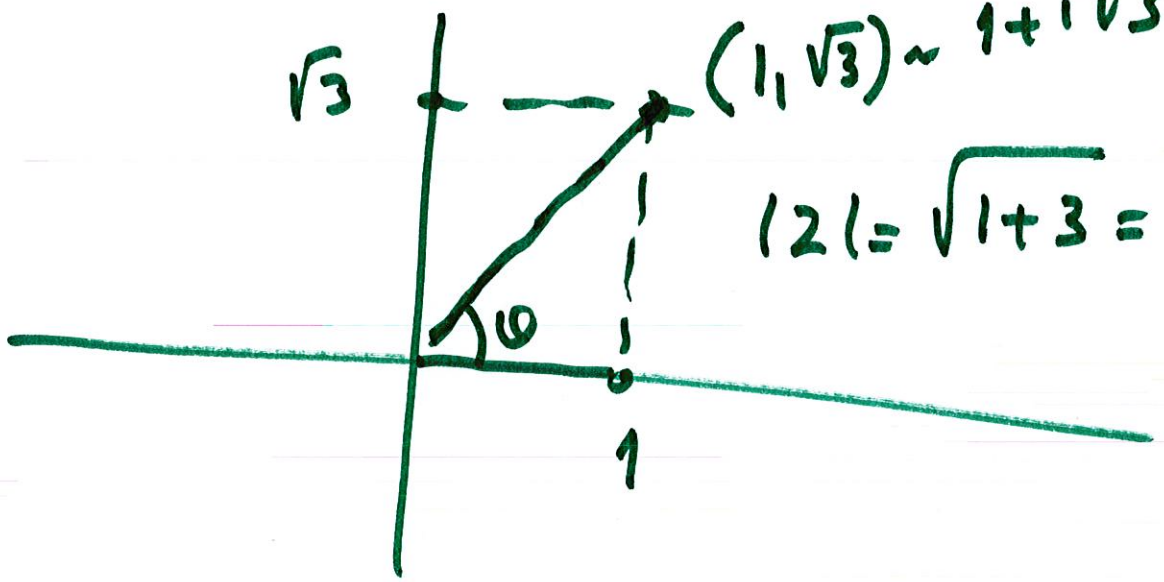
$a = |z| \cos\theta$
$b = |z| \sin\theta$

$$z = a + ib = |z| \cos\theta + i|z| \sin\theta$$

$$z = |z| (\cos\theta + i\sin\theta)$$

$$\boxed{z = |z| e^{i\theta}}$$

$$z = 1 + i\sqrt{3}$$



$$|z| = \sqrt{1+3} = 2$$

$$\cos\theta = 1/2, \quad \sin\theta = \sqrt{3}/2$$

$$\theta = \pi/3$$

$$z = 2\left(\cos\pi/3 + i\sin\pi/3\right)$$

$$\boxed{z = 2e^{i\pi/3}}$$

$$z = r e^{i\theta} \quad , \quad |z| = r.$$

$$z^n = r^n e^{in\theta}$$

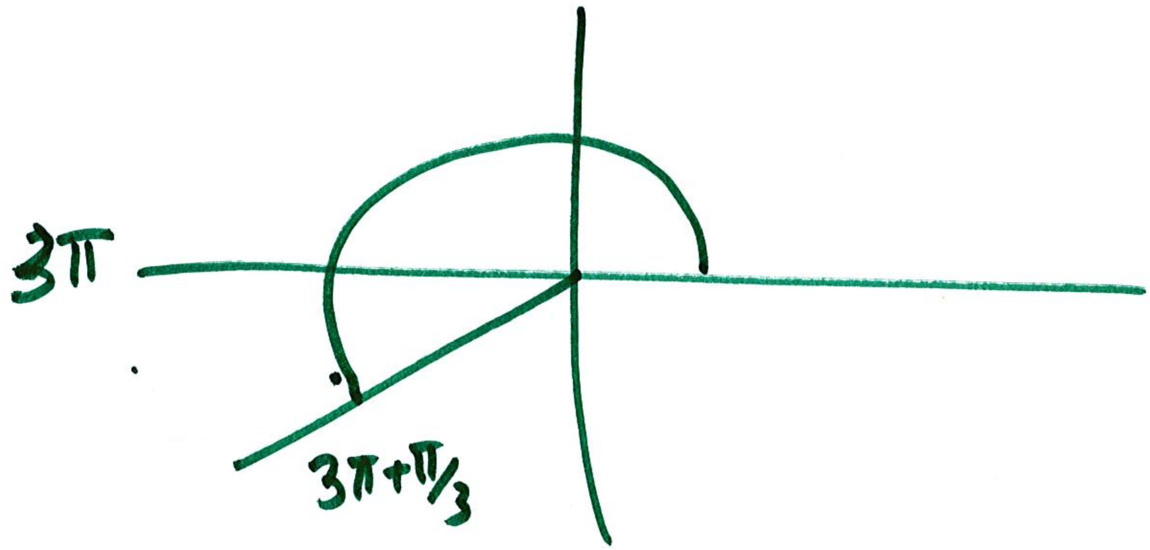
Ex: $z = 1 + i\sqrt{3}$

Compute z^{10}.

$$z = 1 + i\sqrt{3} = 2 e^{i\pi/3}$$

$$z^{10} = 2^{10} e^{i\,10\pi/3}.$$

$$10\pi/3 = 3\pi + \pi/3$$



$$e^{i\,10\pi/3} = \cos 10\pi/3 + i \sin 10\pi/3$$

$$= -1/2 - i\sqrt{3}/2$$

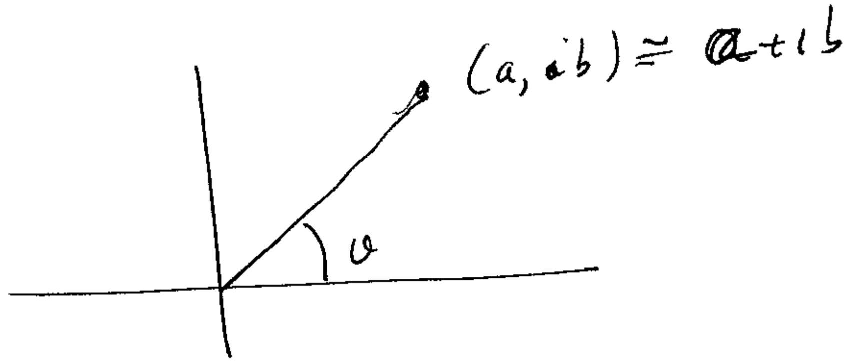
$$z^{10} = 2^{10}\left(-1/2 - i\sqrt{3}/2\right)$$

$$= -2^{9}\left(1 + i\sqrt{3}\right).$$

$$\frac{2+i}{3-4i} = (2+i) \cdot \frac{1}{3-4i} = \frac{(2+i)(3+4i)}{25}$$ ②

$$= \frac{1}{25}(6+8i+3i-4) = \frac{1}{25}(2+11i)$$

Polar representation



$$z = r(\cos\theta + i\sin\theta)$$

Taylor Series:

$i = \sqrt{-1}$, $\quad i^2 = -1$, $\quad i^3 = -i$

$i^4 = 1$. $\quad i^5 = i$

$$i^{2k} = (-1)^k$$
$$i^{2k+1} = (-1)^k i$$