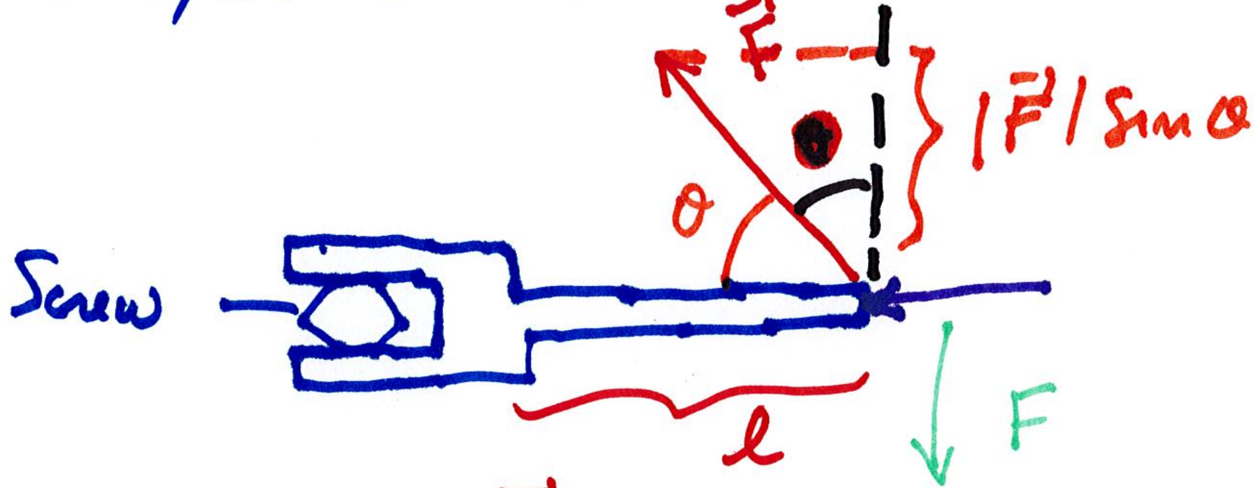
Lesson 4 Section 12.4

The Cross Product

Physical Motivation.

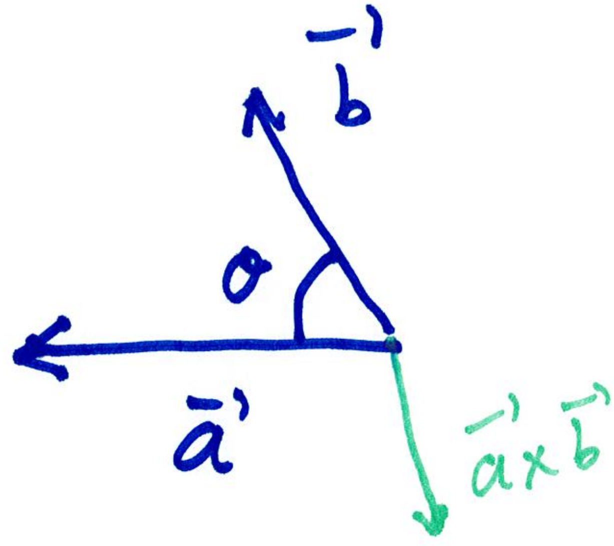


Torque $\vec{\tau}$:

$$|\vec{\tau}| = |\vec{F}| \cdot \sin\theta \cdot \ell.$$

Direction of $\vec{\tau}$ = Given by the right hand rule

$\vec{\tau}$ is perpendicular to $\vec{F}$ and to the wrench.



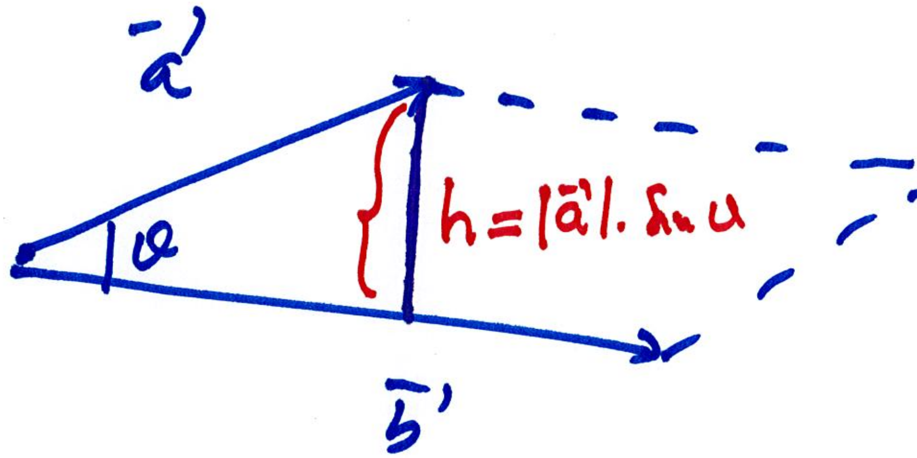
$\vec{a} \times \vec{b}$ is the vector which satisfies:

(1) $|\vec{a} \times \vec{b}| = |\vec{a}| . |\vec{b}| . \sin\theta$.

(2) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

(3) It's direction is given by the right hand rule.

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.}$$



$|\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$ = area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

Formula For the Cross Product In terms of coordinates.

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle.$$

Find a vector

$\rightarrow \vec{c} = \langle c_1, c_2, c_3 \rangle$

which is perpendicular to both.

$\vec{c} \cdot \vec{a} = 0 \qquad a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$

$\vec{c} \cdot \vec{b} = 0 \qquad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$

$\vec{c} = \vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,$

$a_1 b_2 - a_2 b_1 \rangle.$

One way to remember this formula is to use determinants.

$$\det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

$$\det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Example:</u> Find $\langle 1, 2, 1 \rangle \times \langle -1, 2, 3 \rangle$.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{vmatrix} =$$

$$= \vec{i} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}$$

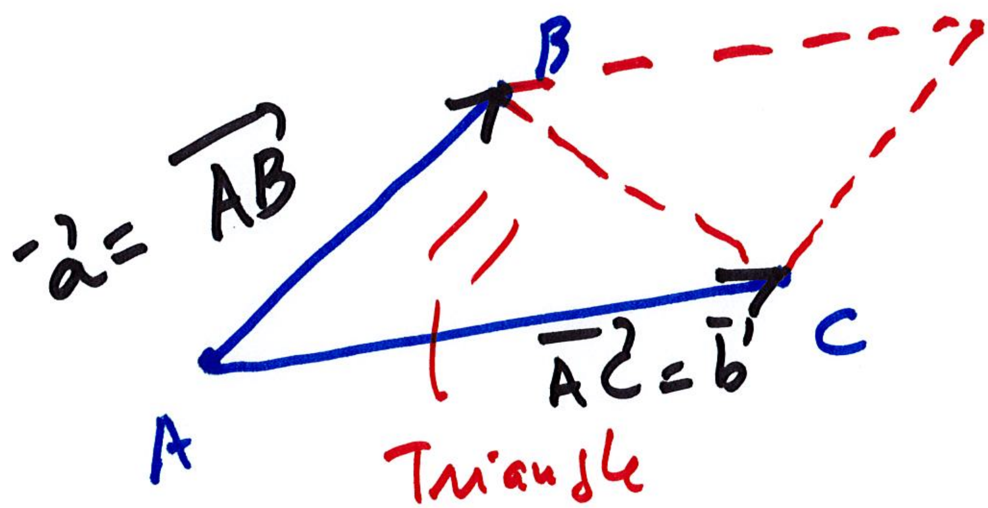
$$+ \vec{k} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= \vec{i}(6-2) - \vec{j}(3-(-1))$$

$$+ \vec{k}(2-(-2))$$

$$= 4\vec{i} - 4\vec{j} + 4\vec{k}.$$

Find the area of the triangle with vertices $A(1,0,1)$, $B(0,1,4)$, $C(3,1,5)$.



Area of the triangle $= \frac{1}{2}$ area of the parallelogram.

$\vec{a} = \overrightarrow{AB}$

$\cancel{\vec{a}\times\vec{b}}$

$\vec{b} = \overrightarrow{AC}$

Area of triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$.

$$\vec{AB} = \langle -1, 1, 3 \rangle = \vec{a}.$$

$$\vec{AC} = \langle 2, 1, 4 \rangle. = \vec{b}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 3 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= \vec{i}\,(1) - \vec{j}\,(-10) + \vec{k}\,(-3)$$

$$= \vec{i} + 10\vec{j} - 3\vec{k}.$$

Area of the triangle $= \frac{1}{2}\sqrt{1 + 100 + 9}$

$$= \frac{1}{2}\sqrt{110}.$$

Question from a student: Where is $\sin\theta$ in all of this?

$$\vec{c} = \vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

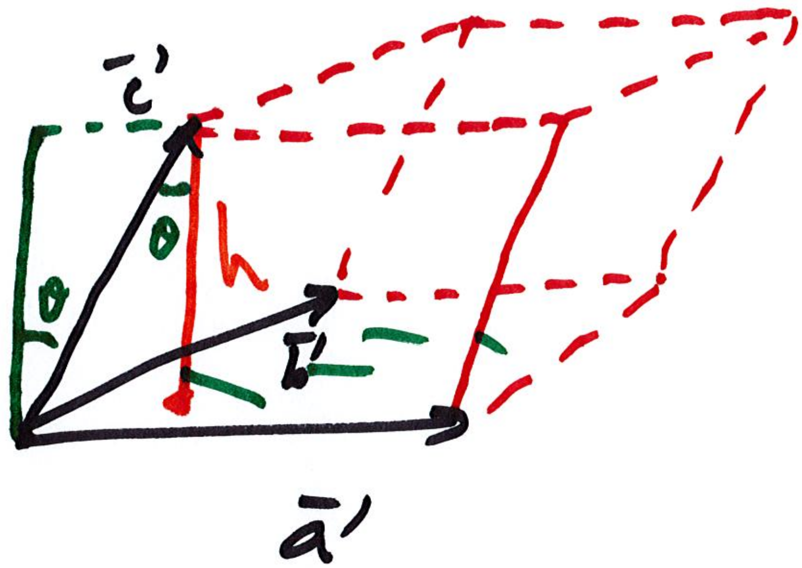
$$|\vec{a} \times \vec{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2$$

$$+ (a_1 b_2 - a_2 b_1)^2.$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 - (\underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3})^2.$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 - (|\vec{a}| \cdot |\vec{b}| \cdot \cos\theta)^2$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 (1 - \cos^2\theta).$$



Compute the volume of this solid.

Area of the base $= |\vec{a} \times \vec{b}|$.

height $h = |\vec{c}| \cdot \cos\theta$.

$$\text{Volume} = \big|\, |\vec{a} \times \vec{b}| \cdot |\vec{c}| \cdot \cos\theta \,\big|$$
$$= \big| (\vec{a} \times \vec{b}) \cdot \vec{c} \,\big|$$

Additional Note

$\vec{a} = \langle a_1, a_2, a_3 \rangle; \quad b = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\vec{c} \cdot \vec{a} \times \vec{b} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example: Compute the Volume of the Solid formed by the vectors:

$$\vec{a} = \langle 1, 2, 3 \rangle, \quad \vec{b} = \langle 0, 1, 5 \rangle$$

and $\vec{c} = \langle -1, 1, 2 \rangle$.

Step 1: $(\vec{a} \times \vec{b}) \cdot \vec{c} =$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 5 \end{vmatrix}$$

$$= -1(10-3) - 1(5-0) + 2(1-0)$$

$$= -7 - 5 + 2 = -10.$$

$$\text{Volume} = |-10| = 10.$$