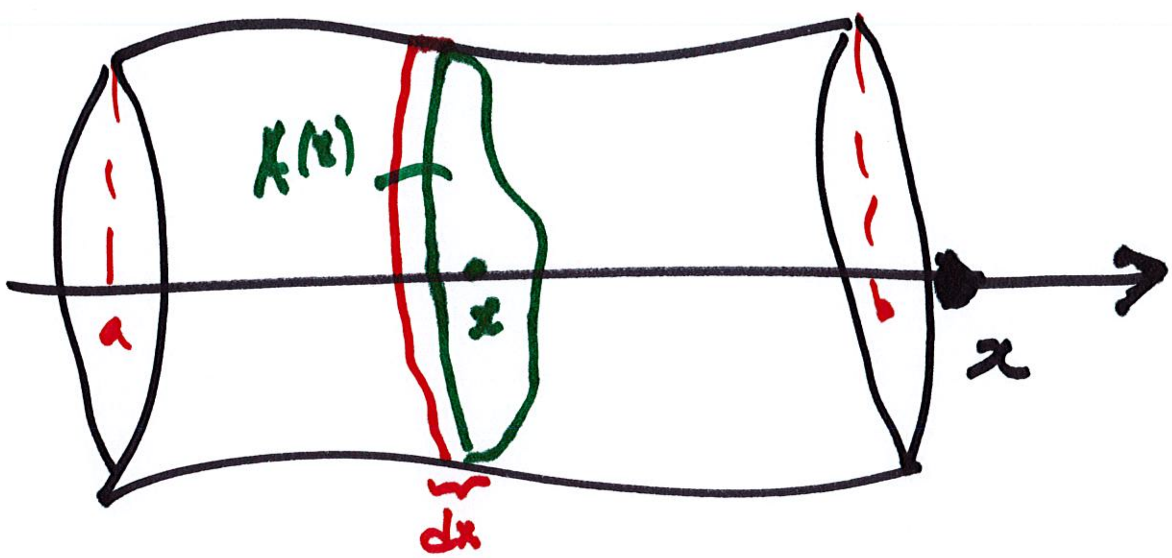
Lesson 6 Section 6.2

Volumes

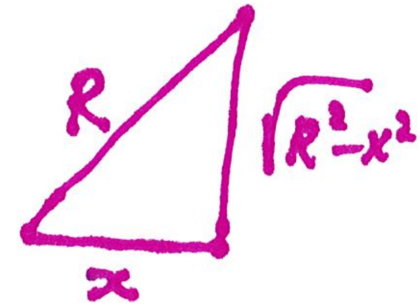


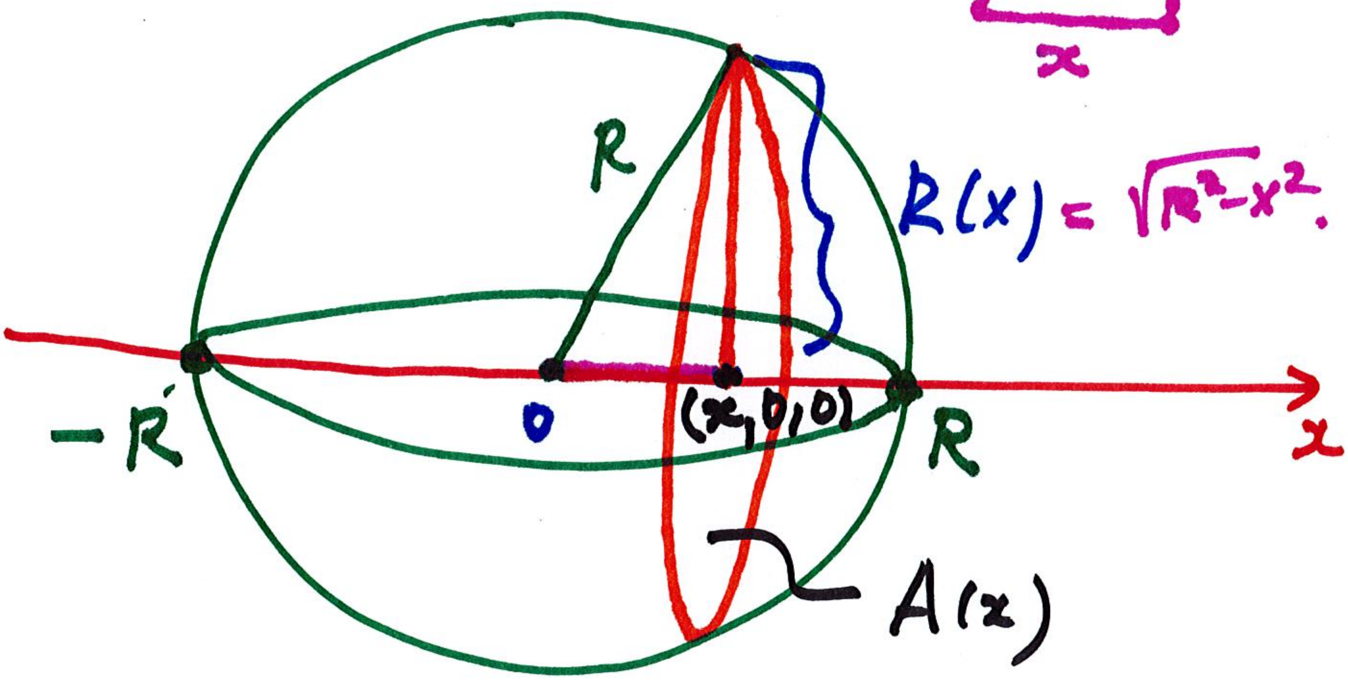
$A(x)$ = area of the cross section of the object.

$dV = A(x)\,dx$ = Volume of a very thin slice.

$$\text{Volume } V = \int_a^b A(x)\,dx$$

Examples





Area of the cross section

$$= \pi R(x)^2$$

Equation of the sphere:

$$x^2 + y^2 + z^2 = R^2$$

$$y^2 + z^2 = R^2 - x^2$$

$$R(x) = \sqrt{R^2 - x^2}$$

$$A(x) = \pi R(x)^2 = \pi (R^2 - x^2).$$

$$V = \int_{-R}^{R} \pi (R^2 - x^2)\, dx$$

$$= \int_{-R}^{R} \pi R^2\, dx - \int_{-R}^{R} \pi x^2\, dx$$

$$= 2\pi R^3 - \frac{\pi}{3} x^3 \Big|_{-R}^{R}$$

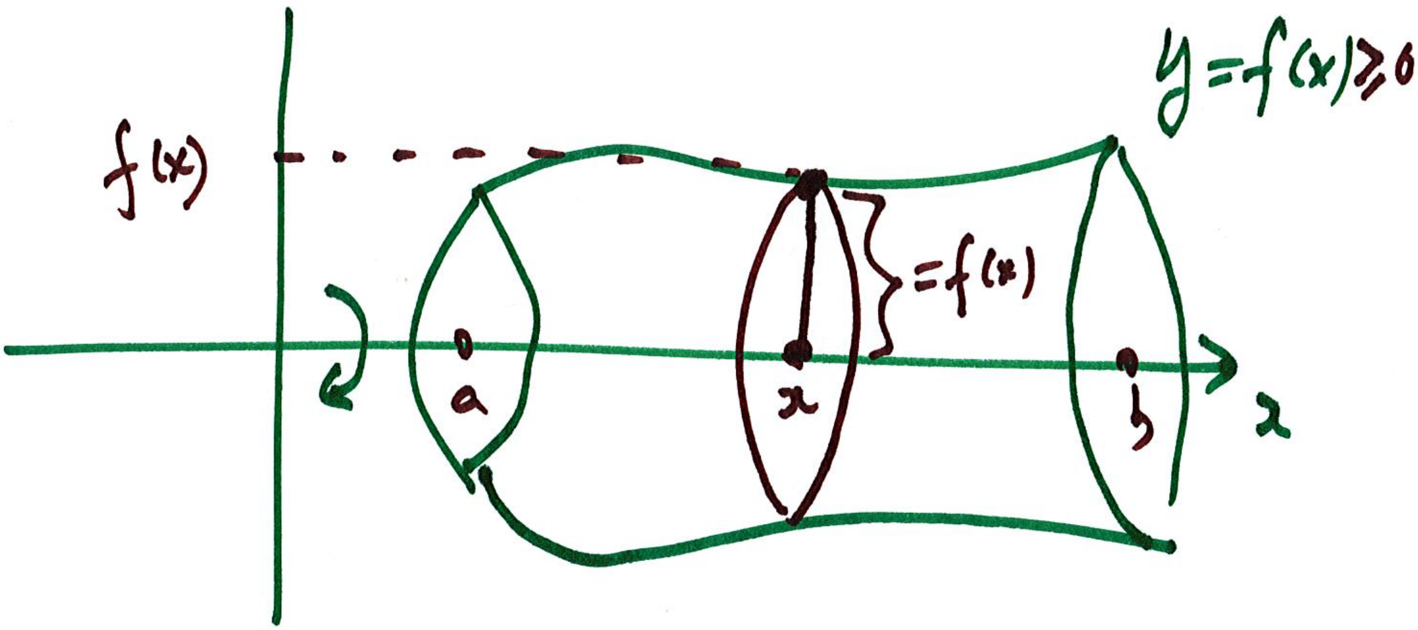
$$= 2\pi R^3 - 2\pi \frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

$$\boxed{\int_{-R}^{R} x^2\, dx = \frac{1}{3} x^3 \Big|_{-R}^{R} = \frac{1}{3}\left(R^3 - (-R)^3\right) = \frac{2R^3}{3}.}$$

$$\int_{-R}^{R} \pi R^2 \, dx = \pi R^2 \int_{-R}^{R} dx$$

$$= \pi R^2 \cdot 2R = 2\pi R^3.$$

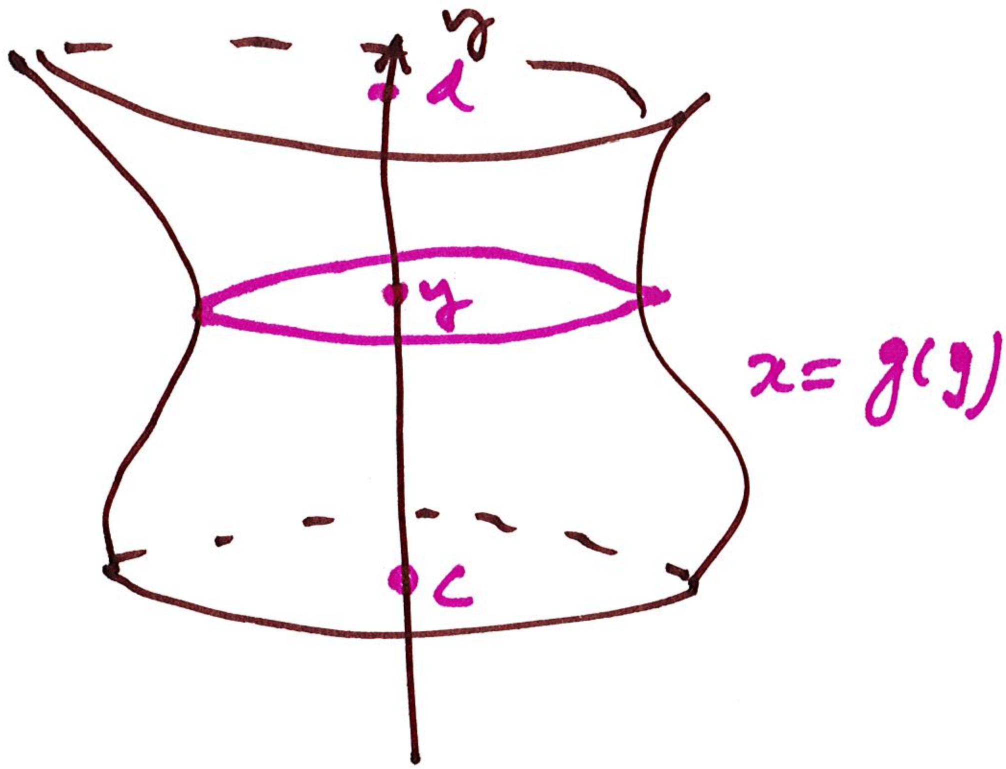
Solids of Revolution



Cross sections are circles.

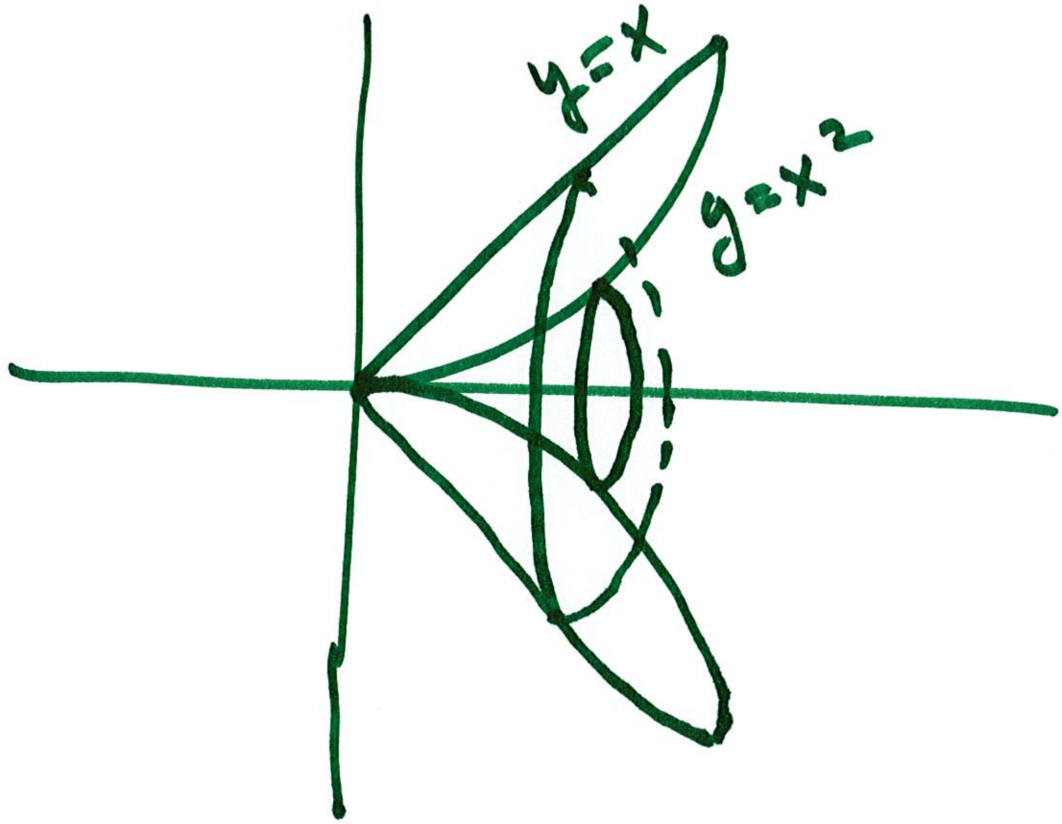
$$A(x) = \pi \cdot (f(x))^2.$$

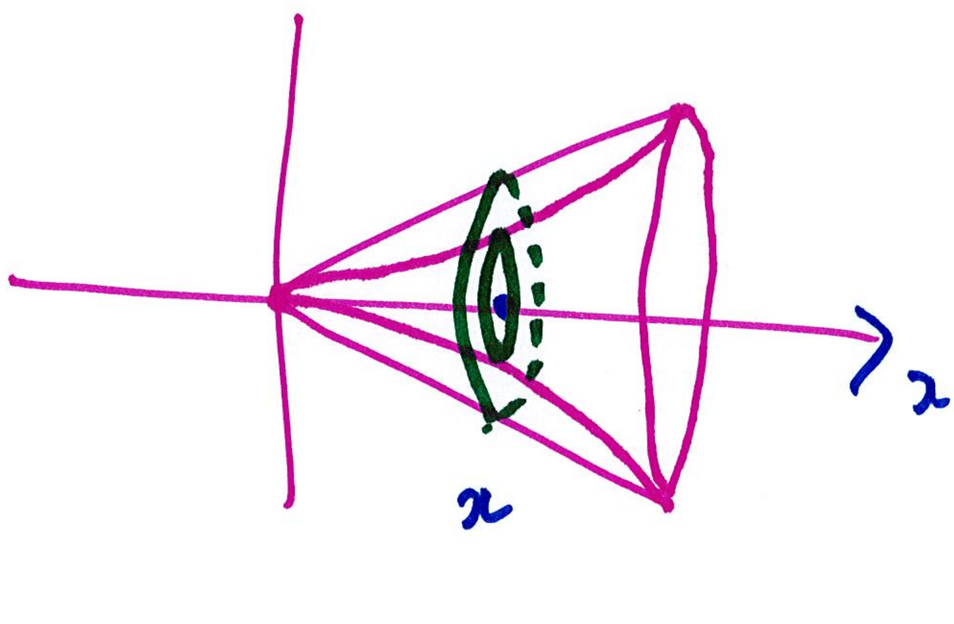
$$\text{Volume} = \int_a^b \pi (f(x))^2 \, dx$$

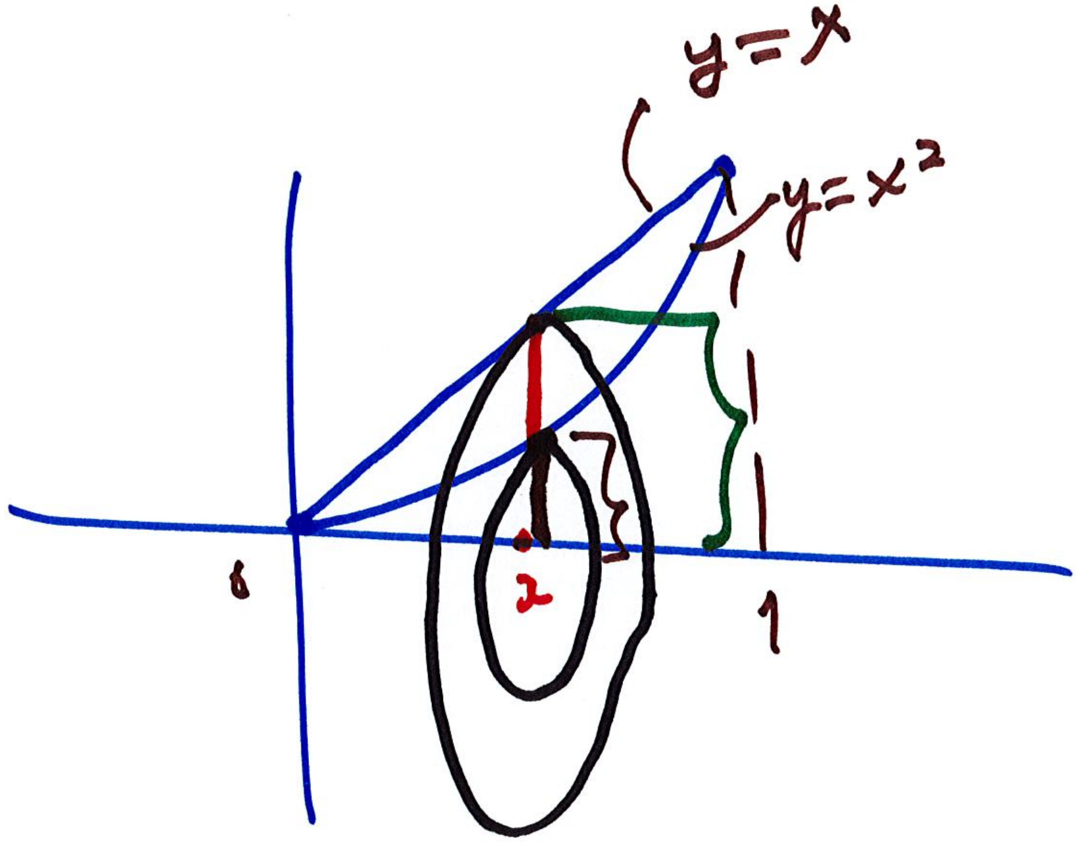


$$A(y) = \pi\,(g(y))^2$$

$$V = \int_c^d \pi\,(g(y))^2\,dy$$


y=x
y=x2


x
x



Outer Circle = $\pi \cdot x^2$

Inner Circle = $\pi \cdot (x^2)^2$

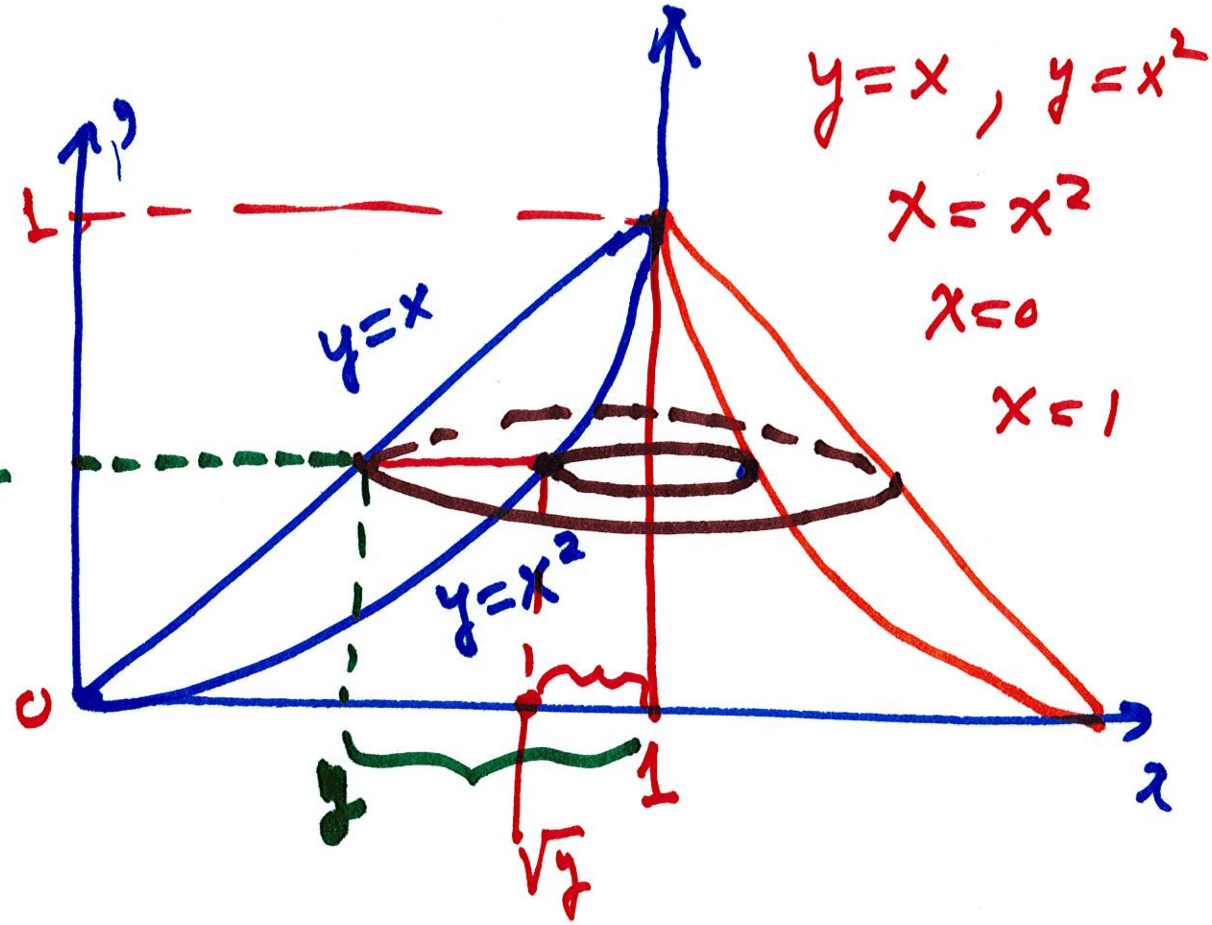
Area of the cross Section

$= \pi\,(x^2 - x^4)$.

$$V = \int_0^1 \pi (x^2 - x^4)\, dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$



Outer radius $= 1 - y$

Inner radius $= 1 - \sqrt{y}$.

Area of the cross section

$$A(y) = \pi (1-y)^2 - \pi (1-\sqrt{y})^2.$$

$$\text{Volume} = \int_0^1 \pi\left((1-y)^2 - (1-\sqrt{y})^2\right)dy$$

$$= \pi \int_0^1 \left(1-2y+y^2 - (1-2\sqrt{y}+y)\right)dy$$

$$= \pi \int_0^1 (2\sqrt{y} - 3y + y^2)\,dx$$

$$= \pi \left(2 \cdot \tfrac{2}{3} y^{3/2} - \tfrac{3}{2} y^2 + \tfrac{y^3}{3}\right)\Big|_0^1$$

$$= \pi\left(4/3 + 1/3 - 3/2\right) = \pi\left(5/3 - 3/2\right)$$

$$= \frac{\pi}{6}$$