Lesson 11 Section 7.3

Trigonometric Substitutions

Part I.

Example: $\displaystyle\int_0^1 \underbrace{\sqrt{1-x^2}}\, dx$

Idea: Make a substitution which transforms this into a trigonometric integral.

$x = \sin\theta$, $\underline{dx = \cos\theta \, d\theta}$

$\cos^2\theta + \sin^2\theta = 1$

$\underline{\cos^2\theta} = \underline{1 - \sin^2\theta}$

$x = \sin\theta$, $\cos^2\theta = 1 - x^2$

$\cos\theta = \sqrt{1-x^2}.$

$x=0$, $\sin\theta = 0$, $\theta = 0$; $x=1$, $\sin\theta = 1$, $\theta = \pi/2$

$$\int_0^1 \underbrace{\sqrt{1-x^2}}_{\cos\theta} \, \underbrace{dx}_{\cos\theta \, d\theta} = \int_0^{\pi/2} \cos\theta . \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \, d\theta.$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_0^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

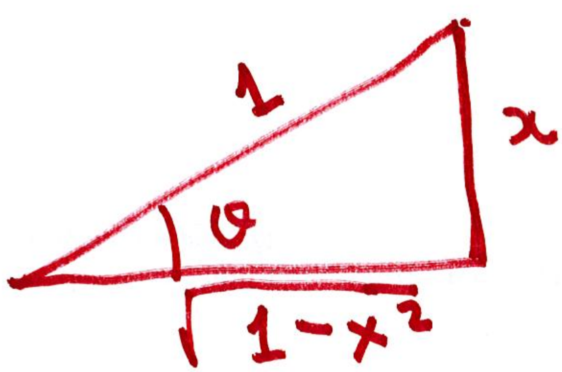
$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

— 〃 —— $\sqrt{1-x^2}$, x

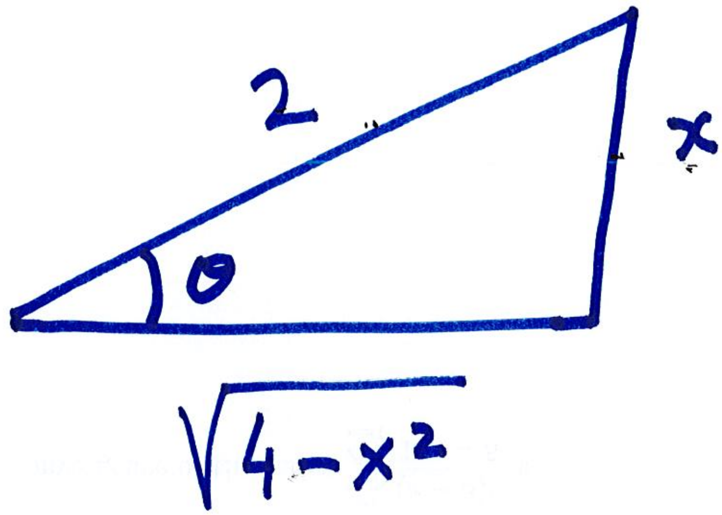
$\sin\theta = x/1$

$\cos\theta = \dfrac{\sqrt{1-x^2}}{1}$.



$$\int x^2 \underbrace{\sqrt{4-x^2}} \, dx$$

Step 1.



$$\sin\theta = x/2 \; ; \quad \boxed{\cos\theta = \frac{\sqrt{4-x^2}}{2}}$$

$$x = 2\sin\theta \quad , \quad dx = 2\cos\theta \, d\theta$$

$$\int x^2 \underbrace{\sqrt{4-x^2}}_{2\cos\theta} \, \underbrace{dx} = \int (2\sin\theta)^2 \cdot 2\cos\theta \, 2\cos\theta \, d\theta$$

$$(\sin\theta \cos\theta)^2$$

$$= 16 \int \sin^2\theta \cos^2\theta \, d\theta .$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\boxed{\sin 2\theta = 2\sin\theta . \cos\theta}$$

$$= \int 16 . \left[\frac{\sin 2\theta}{2}\right]^2 d\theta .$$

$$= 4\int [\sin 2\theta]^2 \, d\theta .$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} .$$

$\alpha = 2\theta$

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}.$$

$$\int 4(\sin 2\theta)^2 \, d\theta =$$

$$4 \int \left(\frac{1-\cos 4\theta}{2} \right) d\theta$$

$$= 2 \int (1-\cos 4\theta) \, d\theta$$

$$= 2\theta - \frac{1}{2}\sin 4\theta + c$$

Write the answer in terms of x.

$$\int x^2 \sqrt{4-x^2}\, dx$$

$$= 2\theta - \frac{1}{2}\sin 4\theta + C.$$

$$\boxed{\sin\theta = x/2}\;; \quad \cos\theta = \frac{1}{2}\sqrt{4-x^2}$$

$$\theta = \arcsin\left(x/2\right) = \sin^{-1}\left(x/2\right)$$

Write $\sin 4\theta$ in terms of x.

$$\sin 4\theta = \sin(2\theta + 2\theta) = \sin(2 \cdot 2\theta)$$

$$= 2 \, \sin 2\theta \, . \, \cos 2\theta \, .$$

$$= 2 \, . \, 2 \sin\theta \cos\theta \, . \left(\cos^2\theta - \sin^2\theta \right)$$

$$= 4 \, \sin\theta \, . \, \cos\theta \left(\cos^2\theta - \sin^2\theta \right).$$

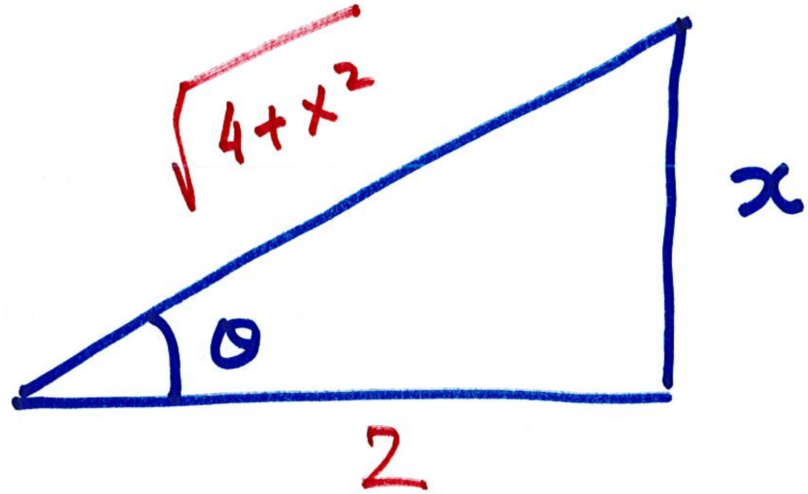
$$= \cancel{4} \, . \, x/\cancel{2} \cdot \frac{1}{\cancel{2}} \sqrt{4-x^2} \, . \left(\frac{4-x^2}{4} - \frac{x^2}{4} \right)$$

$$= \frac{x}{4} \sqrt{4-x^2} \, \left(4 - 2x^2\right)$$

<u>Conclusion:</u> $\int x^2 \sqrt{4-x^2} \, dx =$

$$= 2 \sin^{-1}(x/2) - \frac{x}{8} \sqrt{4-x^2} \left(4-2x^2\right) + C$$

$$\int \frac{dx}{\sqrt{x^2+4}}$$



$$x/2 = \tan\theta \; ; \quad \frac{2}{\sqrt{4+x^2}} = \cos\theta$$

$$\boxed{\sqrt{4+x^2} = \frac{2}{\cos\theta} = 2\sec\theta}$$

$x = 2\tan\theta$, $\quad dx = 2\sec^2\theta \, d\theta$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Recall that

$$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$$

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta.$$

$$\tan\theta = x/2 \ ; \quad \sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

$$\sec\theta = \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$$

$$= \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}$$

Set $u = \sec\theta + \tan\theta$

$$\int \sec\theta \, d\theta = \int \frac{du}{u} = \ln|u| + c$$

$$= \ln|\sec\theta + \tan\theta| + c.$$

$$\int \frac{dx}{\sqrt{4+x^2}} =$$

$$= \ln \left| \frac{x}{2} + \frac{1}{2}\sqrt{4+x^2} \right| + C.$$

$$\int \sec\theta \cdot \tan^2\theta \, d\theta. =$$

$$= \int \underbrace{\tan\theta}_{u} \cdot \underbrace{\sec\theta \tan\theta \, d\theta}_{dv}$$

$$= uv - \int v \, du$$

$$V = \sec\theta.$$

$$du = \sec^2\theta \, d\theta$$

$$\underline{\int \tan\theta \cdot \sec\theta \tan\theta \, d\theta} =$$

$$= \tan\theta \cdot \sec\theta - \int \sec^3\theta \, d\theta$$

$$\sec^3\theta = \sec\theta \left(\sec^2\theta \right)$$

$$= \sec\theta \left(1 + \tan^2\theta \right)$$

$$= \tan\theta \cdot \sec\theta - \int \sec\theta \, d\theta - \underline{\int \sec\theta \tan^2\theta}$$

$$2\int \sec\theta \tan^2\theta \, d\theta$$

$$= \sec\theta \tan\theta - \int \sec\theta \, d\theta.$$

$$\int \sec\theta \tan^2\theta \, d\theta$$

$$= \frac{1}{2} \sec\theta \tan\theta - \frac{1}{2} \ln|\sec\theta + \tan\theta|$$

$$+ C$$