Lesson 12 . Section 7.3

Trigonometric Substitution Part II

Last time:

Integrals involving $\sqrt{a^2-x^2}$

and $\sqrt{a^2+x^2}$.

In the case $\sqrt{a^2-x^2}$; $x = a\sin\theta$

$\boxed{\sqrt{a^2+x^2},}$ ~~$x = a\tan\theta$~~

$x = a\tan\theta$.

$$\int_{0}^{2} x^3 \sqrt{4+x^2}\, dx$$

$\sqrt{a^2+x^2}$, $a^2=4$, $a=2$.

Set $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$4+x^2 = 4+4\tan^2\theta$

$= 4(\underbrace{1+\tan^2\theta}_{\sec^2\theta}) = 4\sec^2\theta$

$$\boxed{\sqrt{4+x^2} = 2\sec\theta}$$

$x=0$; $2\tan\theta=0$, $\theta=0$

$x=2$, $2\tan\theta=2$, $\theta=\pi/4$

$$\int_0^2 x^3 \sqrt{4+x^2}\, dx$$

$$= \int_0^{\pi/4} (2\tan\theta)^3 \cdot 2\sec\theta \cdot 2\sec^2\theta\, d\theta$$

$$= 32 \int_0^{\pi/4} \underline{\tan^3\theta}\ \underline{\sec^3\theta}\, d\theta$$

$$\frac{d}{d\theta} \sec\theta = \sec\theta \tan\theta$$

$$= 32 \int_0^{\pi/4} \underbrace{\tan^2\theta}_{=\sec^2\theta - 1} \cdot \cancel{\tan}\sec^2\theta \cdot \sec\theta \tan\theta\, d\theta$$

$$= 32 \int_0^{\pi/4} (\sec^2\theta - 1)\sec^2\theta \cdot \overbrace{\sec\theta \tan\theta\, d\theta}^{du}$$

$$\boxed{u = \sec\theta} \qquad du = \sec\theta \tan\theta\, d\theta$$

$$= \int_1^{\sqrt{2}} (u^2 - 1) u^2 \, du.$$

$$\sec \pi/4 = \frac{1}{\cos \pi/4} = \frac{1}{1/\sqrt{2}} = \sqrt{2}.$$

$$= \int_1^{\sqrt{2}} u^2 (u^2 - 1) \, du.$$

Variation of this problem

$$\int_{-1}^{2} x\sqrt{x^2+2x+10}\,dx$$

$$x^2+2x+10 = (x+1)^2-1+10$$

$$= (x+1)^2+9.$$

$$\int_{0}^{2} x\,\overbrace{\sqrt{\underbrace{(x+1)}^{}{}^2+9}}^{=3\sec\alpha}\,dx.$$

$$x+1 = 3\tan\alpha.$$

$$dx = 3\sec^2\alpha\,d\alpha.$$

$$\int \frac{dx}{\sqrt{(x-3)^2-4}} =$$

$x-3 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$$\int \frac{2\sec\theta\tan\theta}{2\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta.$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C.$$

$$= \ln\left|\frac{x-3}{2} + \frac{1}{2}\sqrt{(x-3)^2-4}\right| + C.$$

$$\int_1^2 x^3 \sqrt{x^2-1}\, dx$$

$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$=$

$$= \int_0^{\pi/3} \sec^3\theta \cdot \tan\theta \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int_0^{\pi/3} \sec^4\theta \cdot \tan^2\theta \, d\theta .$$

$$= \int_0^{\pi/3} \sec^2\theta \cdot \sec^2\theta \cdot \tan^2\theta \, d\theta$$

$$= \int_0^{\pi/3} \sec^2\theta \left(\underline{\tan^2\theta + 1}\right) \underline{\tan^2\theta} \, d\theta .$$

$u = \tan\theta .$

$$\int_0^{\sqrt{3}} u^2(u^2+1)\,du.$$

$$\int \frac{dx}{\sqrt{x^2-6x+5}}$$

Step 1: Complete the square

$$x^2-6x+5 = (x-3)^2-9+5$$

$$= (x-3)^2-4.$$

$$\int \frac{dx}{\sqrt{(x-3)^2-4}}.$$

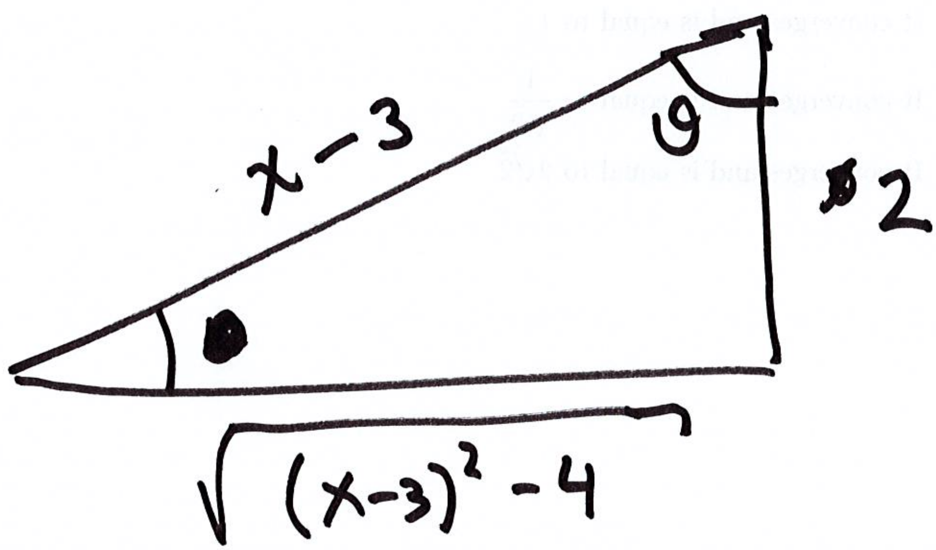
$$-3 + x = 2\sec\theta.$$

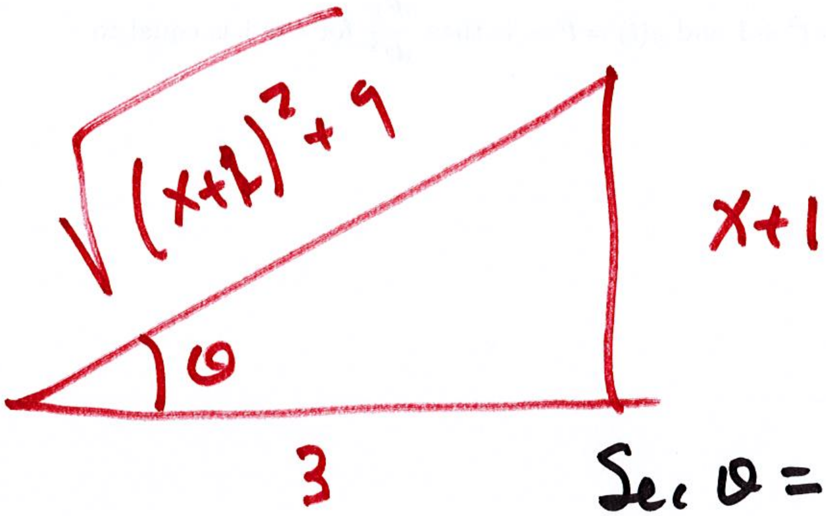
$$(x-3)^2 - 4 = 4\sec^2\theta - 4$$

$$= 4(\sec^2\theta - 1)$$

$$= 4\tan^2\theta$$

$$\sqrt{(x-3)^2 - 4} = 2\tan\theta.$$





$$\sec\theta = \frac{\sqrt{(x+1)^2+9}}{3}$$

$$= \int_0^{\pi/4} (3\tan\theta - 1) \cdot 3\sec\theta \cdot 3\sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} 27\tan\theta\sec\theta \cdot \sec^2\theta \, d\theta$$

$$- 9\int_0^{\pi/4} \sec^3\theta \, d\theta .$$

$$27\int_0^{\pi/4} \underline{\tan\theta \cdot \sec\theta} \cdot \sec^2\theta \, d\theta =$$

$$\sec\theta = u$$

$$du = \tan\theta \sec\theta \, d\theta$$

$$= 27\int_1^{\sqrt{2}} u^2 \, du$$

$$\int \sec^3\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta =$$

$$= \int \sec\theta \cdot \left(\underbrace{\sec^2\theta}_{\tan^2\theta + 1} \right) d\theta$$

$$= \int \sec\theta \left(1 + \tan^2\theta \right) d\theta$$

$$= \int \sec\theta \, d\theta + \underbrace{\int \sec\theta \tan^2\theta \, d\theta}$$

$$\int \sec\theta \, d\theta = \underline{\ln|\sec\theta + \tan\theta|} + C.$$

$$\int \sec\theta \tan^2\theta \, d\theta.$$

$$= \int \underbrace{\tan\theta}_{u} . \underbrace{\sec\theta \tan\theta \, d\theta}_{dv}$$

$u = \underline{\underline{\tan\theta}}$; $dv = \sec\theta \tan\theta \, d\theta$

$du = \sec^2\theta \, d\theta$ $\quad V = \underline{\underline{\sec\theta}}$

$$= \tan\theta . \sec\theta - \int \sec^3\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta =$$

$$= \ln|\sec\theta + \tan\theta| + \tan\theta \sec\theta$$

$$- \int \sec^3\theta \, d\theta$$

$$2\int \sec^3\theta \, d\theta = \ln|\sec\theta + \tan\theta|$$

$$+ \tan\theta \sec\theta + C$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$$+ \frac{1}{2}\tan\theta \sec\theta + C.$$

$$\int_0^1 \sqrt{1+x^2}\, dx$$

$x = \tan\theta, \quad dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/4} \sec\theta \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta.$$