Lesson 13 Section 7.4

Partial Fractions

$$\int \frac{dx}{(x-2)(x-1)} dx = I$$

We know how to compute

$$I_1 = \int \frac{dx}{x-2} \quad \text{and} \quad \int \frac{dx}{x-1} = I_2$$

Can we "reduce" I to

I_1 and I_2

Question: Can we write

$$\frac{1}{(x-1)(x-2)} \stackrel{?}{=} \frac{A}{x-1} + \frac{B}{x-2}$$

A and B are numbers and do not depend on x

If this is the case

$$\int \frac{dx}{(x-1)(x-2)} = A\int \frac{dx}{x-1} + B\int \frac{dx}{x-2}$$

$$= A \ln|x-1| + B \ln|x-2| + C.$$

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\frac{1}{(x-1)(x-2)} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

$$\boxed{1 = \underline{\underline{A}}(x-2) + \underline{\underline{B}}(x-1)}$$

for all $\underline{\underline{x}}$.

Method 1

$1 = Ax - 2A + Bx - B$

$1 = (A+B)x - 2A - B$

$$\begin{cases} A + B = 0 \\ -2A - B = 1. \end{cases}$$

$-A = 1$

$A = -1$

$B = 1$

$A = -1, \quad B = 1$

$A + B = 0$

$-2A - B = 2 - 1 = 1$

Conclusion:

$$\frac{1}{(x-1)(x-2)} = -\frac{1}{x-1} + \frac{1}{x-2}$$

Method 2

$1 = (A+B)x - 2A - B$

$$\boxed{1 = A(x-2) + B(x-1)}$$

If this holds for two values of x, then it holds for every x.

$$1 = A(x-2) + B(x-1)$$

Set $x=1$

$$1 = -A$$

Set $x=2$

$$1 = B$$

Conclusion:

$$\int \frac{dx}{(x-1)(x-2)} = \int \left[\frac{1}{x-2} - \frac{1}{x-1} \right] dx$$

$$= \ln|x-2| - \ln|x-1| + C$$

$$\int \frac{x^2+2x+1}{(x-1)(x-2)(x-3)}\,dx.$$

We want to write

$$\frac{x^2+2x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

We want:

$$A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

$$= x^2+2x+1.$$

$\underline{x=1}$. $2A = 4$

$$\boxed{A=2}$$

$x=2$: $-B = 9$

$$\boxed{B=-9}$$

$\underline{x=3}$: $2C = 16$

$C = 8$

$$\int \frac{x^2+2x+1}{(x-1)(x-2)(x-3)}\,dx =$$

$$= \int \left[\frac{2}{x-1} - \frac{9}{x-2} + \frac{8}{x-3} \right] dx$$

$$= 2\ln|x-1| - 9\ln|x-2| + 8\ln|x-3| + C.$$

$$\int \frac{x+1}{(x-1)(x-2)^2}\,dx.$$

$$\frac{x+1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{\alpha x+\beta}{(x-2)^2}$$

$$= \frac{A}{x-1} + \frac{\alpha(x-2+2)+\beta}{(x-2)^2}$$

$$= \frac{A}{x-1} + \frac{\alpha(x-2) + 2\alpha+\beta}{(x-2)^2}$$

$$= \frac{A}{x-1} + \frac{\alpha}{x-2} + \frac{2\alpha+\beta}{(x-2)^2}.$$

$$\frac{x+1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$B = \alpha, \quad C = 2\alpha + \beta$$

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} =$$

$$= \frac{A(x-2)^2 + B(x-1)(x-2) + C(x-1)}{(x-1)(x-2)^2}$$

<u>We need</u>

$$A(x-2)^2 + B(x-1)(x-2) + C(x-1) = x+1.$$

$C = 3$.

Set $\underline{x=2}$. $\quad C = 3$,

$x=1 \quad \boxed{A = 2}$

$\underline{x=0}$:

$$4A + 2B - C = 1$$

$$8 + 2B - 3 = 1$$

$$2B = -4 \qquad B = -2.$$

$$\int \frac{x+1}{(x-1)(x-2)^2}\,dx =$$

$$\int \left[\frac{2}{x-1} - \frac{2}{x-2} + \frac{3}{(x-2)^2} \right] dx$$

$$= 2\ln|x-1| - 2\ln|x-2| - \frac{3}{x-2} + C.$$

$\underline{x=1}$ $4 = 5A + B + C$

$$B + C = 4 - \frac{15}{4} = \frac{1}{4}$$

$$B + C = \frac{1}{4}$$

$\underline{x=-1}$:

$$2 = 5A + B - C$$

$$B - C = 2 - \frac{15}{4} = -\frac{7}{4}$$

$$B + C = \frac{1}{4} \qquad \boxed{C = 1}$$

$$B - C = -\frac{7}{4}.$$

$$2B = -\frac{6}{4}, \quad 2B = -\frac{3}{2}, \quad \boxed{B = -\frac{3}{4}}$$

$$\int \frac{x+3}{x(x^2+4)}\, dx .$$

$$\frac{x+3}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\boxed{\frac{x+3}{x(x^2+4)} = \frac{3}{4x} - \frac{3}{4}\frac{x}{x^2+4} + \frac{1}{x^2+4}}$$

$$= \frac{A(x^2+4) + x(Bx+C)}{x(x^2+4)}$$

$$x+3 = A(x^2+4) + Bx^2 + Cx .$$

$\underline{x=0}$: $3 = 4A$; $\boxed{A = 3/4 .}$