Lesson 14 Section 7.4

Partial Fractions, Part II

Last time.

$$\int \frac{dx}{(x-1)(x-2)(x-3)}$$

The partial fractions decomposition

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Same strategy applies

$$\frac{x^2+3x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

If we have

$$\frac{x^3 + x^2 + 3x + 1}{(x-1)(x-2)(x-3)} =$$

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} =$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\cancel{=} \frac{x^3 + x^2 + 3x + 1}{(x-1)(x-2)(x-3)}$$

Before we do the Partial Fractions decomposition we have to divide the polynomials.

$$\frac{x^3 + x^2 + 3x + 1}{(x-1)(x-2)(x-3)} =$$

$$(x-1)(x-2)(x-3) =$$

$$= (x-1)(x^2 - 5x + 6)$$

$$= x^3 - 5x^2 + 6x - x^2 + 5x - 6$$

$$= x^3 - 6x^2 + 11x - 6$$

$$\begin{array}{r|l} x^3 + x^2 + 3x + 1 & x^3 - 6x^2 + 11x - 6 \\ \hline -(x^3 - 6x^2 + 11x - 6) & 1 \\ \hline 7x^2 - 8x + 7 & \end{array}$$

$$\frac{x^3 + x^2 + 3x + 1}{x^3 - 6x^2 + 11x - 6} = 1 + \frac{7x^2 - 8x + 7}{x^3 - 6x^2 + 11x - 6}$$

$$\frac{\boxed{7x^2 - 8x + 7}}{\underbrace{x^3 - 6x^2 + 11x - 6}_{= (x-1)(x-2)(x-3)}} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$= \frac{\boxed{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}}{(x-1)(x-2)(x-3)}.$$

We want

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) =$$

$$7x^2 - 8x + 7.$$

<u>Set $x=1$</u>: $2A = 7 + 7 - 8 = 6$

$$\boxed{A = 3}$$

<u>$x=2$</u>:

$$-B = 28 - 16 + 7 = 19$$

$$\boxed{B = -19}$$

<u>$x=3$</u>: $2C = 63 - 24 + 7 = 39$

$$C = \frac{46}{2} = 23.$$

Conclusion:

$$\frac{x^3 + x^2 + 3x + 1}{x^3 - 6x^2 + 11x - 6} =$$

$$1 + \frac{3}{x-1} - \frac{19}{x-2} + \frac{23}{x-3}$$

$$\int \frac{x^3 + x^2 + 3x + 1}{(x-1)(x-2)(x-3)} \, dx$$

$$= x + 3 \ln|x-1| - 19 \ln|x-2|$$

$$+ 23 \ln|x-3| + C.$$

Repeated Factors

$$\int \frac{dx}{(x-1)(x-2)^3}.$$

$$\frac{1}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$+ \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Same strategy works for

$$\frac{x^3+5x^2+6x}{(x-1)(x-2)^3}.$$

However if we have

$$\frac{5x^4+9x^3+7x+1}{(x-1)(x-2)^3}$$

$$= 5 + \frac{\alpha x^3+\beta x^2+\delta x+\theta}{(x-1)(x-2)^3}$$

Then one can use partial fractions to decompose

$$\int \frac{dx}{x(x^2+1)}$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Compute

$$\int_1^2 \frac{dx}{x(x^2+1)^2}.$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^2+1)^2 + x(Bx+C)(x^2+1) + x(Dx+E)}{x(x^2+1)^2}$$

$$A(x^2+1)^2 + x(x^2+1)(Bx+\cancel{C})$$

$$+ \; x(Dx+E) = 1.$$

Set $x=0$ $\qquad A = 1.$

Look for terms of odd powers of x.

x^3: $\quad Cx^3, \quad C = 0.$

x: $\quad xE, \quad E = 0.$

$$(x^2+1)^2 + Bx^2(x^2+1) + Dx^2 = 1.$$

$\underline{x=1}$:

$$4 + 2B + D = 1$$

$$2B + D = -3$$

$x = 2$:

$$25 + 20B + 4D = 1$$

$$20B + 4D = -24$$

$$5B + D = -6$$

$$2B + D = -3$$

$$5B + D = -6$$