Lesson 16 Improper Integrals

Section 7.8

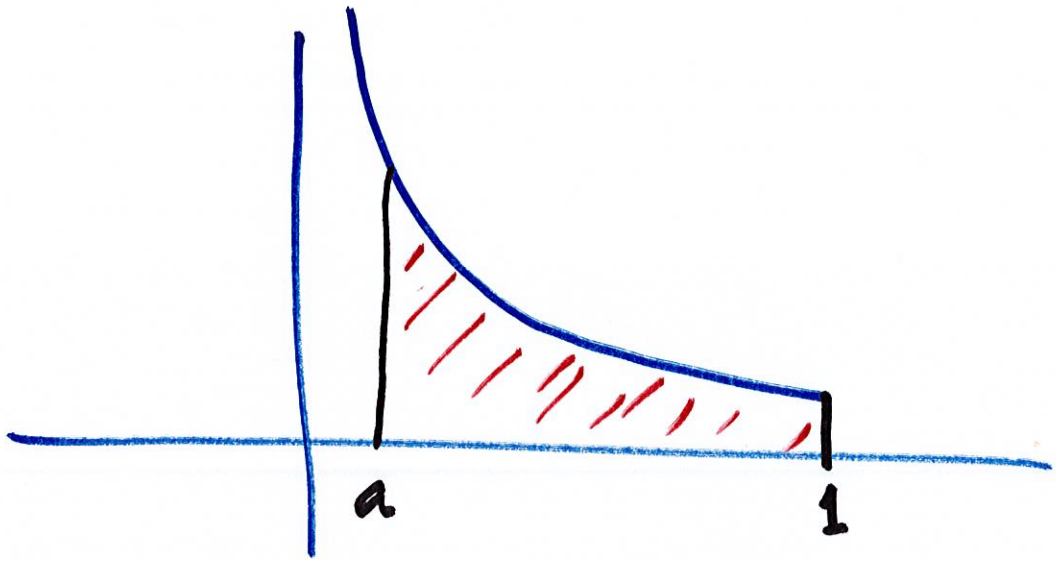
The Fundamental theorem of Calculus:

If $f(x)$ is continuous on $[a, b]$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F'(x) = f(x).$$

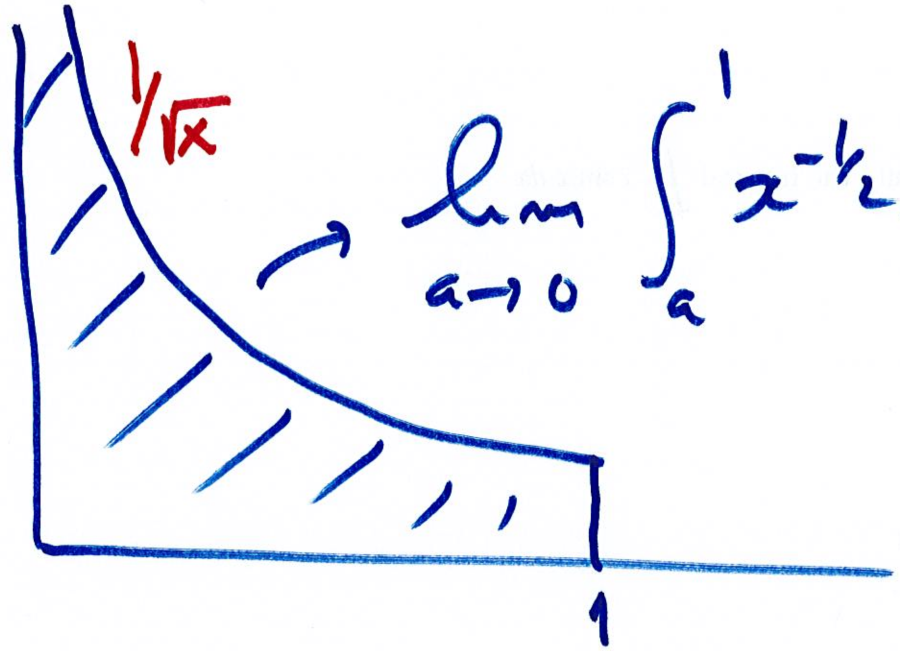
Example: $f(x) = x^{-1/2}, \ x > 0$



When $x \to 0$, $x^{-1/2} = \frac{1}{\sqrt{x}} \to \infty$

$$\int_a^1 x^{-1/2}\,dx$$ = Area of the region above.

$$\lim_{a \to 0} \int_a^1 x^{-1/2}\,dx$$



$$\to \lim_{a\to 0}\int_a^1 x^{-1/2}\,dx$$

$$\lim_{a\to 0}\int_a^1 x^{-1/2} =$$

$$\lim_{a\to 0}\left[\,2x^{1/2}\Big|_a^1\,\right]$$

$$= \lim_{a\to 0}\left(2-2\sqrt{a}\right) = 2.$$

Definition:

$$\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0}\int_a^1 x^{-1/2}\,dx$$

This is called an IMPROPER INTEGRAL.

In this case

$$\int_0^1 x^{-1/2}\,dx = 2$$

We say the integral converges.

$$\int_0^1 x^{-2}\,dx = \lim_{a \to 0} \int_a^1 x^{-2}\,dx$$

$$\int_a^1 x^{-2}\,dx = -\frac{1}{x}\Big|_a^1$$

$$= -\left[1 - \frac{1}{a}\right] = \frac{1}{a} - 1$$

$$\int_a^1 x^{-2}\,dx = -1 + \frac{1}{a}$$

$$\lim_{a \to 0^+} \int_a^1 x^{-2}\,dx = \infty$$

$\int_0^1 x^{-2}\,dx$ diverges

Question: For what values of p does

$$\int_0^1 x^{-p}\, dx$$

converge?

When $p = 1/2$ converges, $p = 2$, diverges.

Case 1: $p = 1$

$$\int_0^1 x^{-1}\, dx = \lim_{a \to 0} \int_a^1 \frac{1}{x}\, dx$$

$$= \lim_{a \to 0} \left[\ln|x| \Big|_a^1 \right] =$$

$$= \lim_{a \to 0} \left[\ln 1 - \ln a \right]$$

$$= \lim_{a \to 0} - \ln a = \lim_{a \to 0} \left(\ln \frac{1}{a} \right)$$

$$= \infty$$

$$\int_0^1 x^{-1}\, dx \text{ diverges.}$$

<u>Case 2</u> : <u>$p \neq 1$</u>

$$\int_a^1 x^{-p}\, dx = \left. \frac{x^{1-p}}{1-p} \right|_a^1$$

$$= \frac{1}{1-p} \left(1 - a^{1-p} \right)$$

$$\lim_{a\to 0} a^{1-p} = \begin{cases} 0 \text{ if } 1-p>0 \\ \infty \text{ if } 1-p<0 \end{cases}$$

Conclusion:

$$\lim_{a\to 0}\int_a^1 x^{-p}\,dx = \frac{1}{1-p} \text{ if } 1-p>0$$

$$\lim_{a\to 0}\int_a^1 x^{-p}\,dx = \infty \text{ if } 1-p<0$$

$\int_0^1 x^{-p}\,dx$ converges if $p<1$

diverges if $p\geq 1$.

Find my mistake

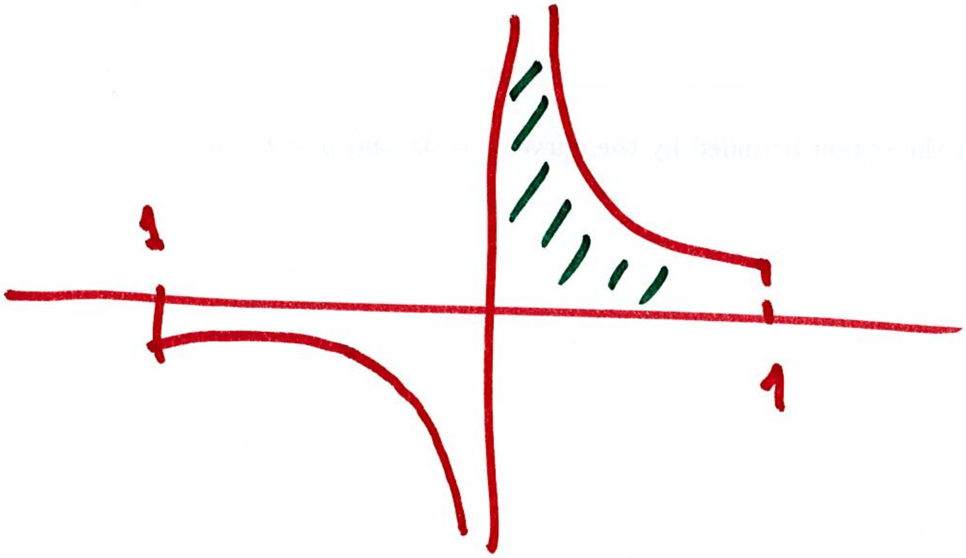
$$\int_{-1}^{1} \frac{1}{x}\,dx \overset{\times}{=} \ln|x|\Big|_{-1}^{1}$$

$$= \ln 1 - \ln|-1|$$

$$= \ln 1 - \ln 1 = 0$$

$$\boxed{\int_{-1}^{1} \frac{1}{x}\,dx = 0}$$

The mistake is that $1/x$ is not a continuous function.



Comparison test for Improper Integrals.

Suppose $f(x) \geq g(x) \geq 0$

on $(a, b]$

$f(x)$ and $g(x)$ continuous on $(a, b]$.

If

$$\int_a^b f(x)\,dx < \infty$$

then $\int_a^b g(x)\,dx$ also Converses.

$$f(x) \geq g(x) \geq 0$$

So

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

If $\int_a^b g(x)\,dx = \infty$, then

$$\int_a^b f(x)\,dx = \infty$$

Decide whether

$$\int_0^1 \frac{1}{x+\sqrt{x}}\,dx$$

Converges or not.

for $x \geq 0$

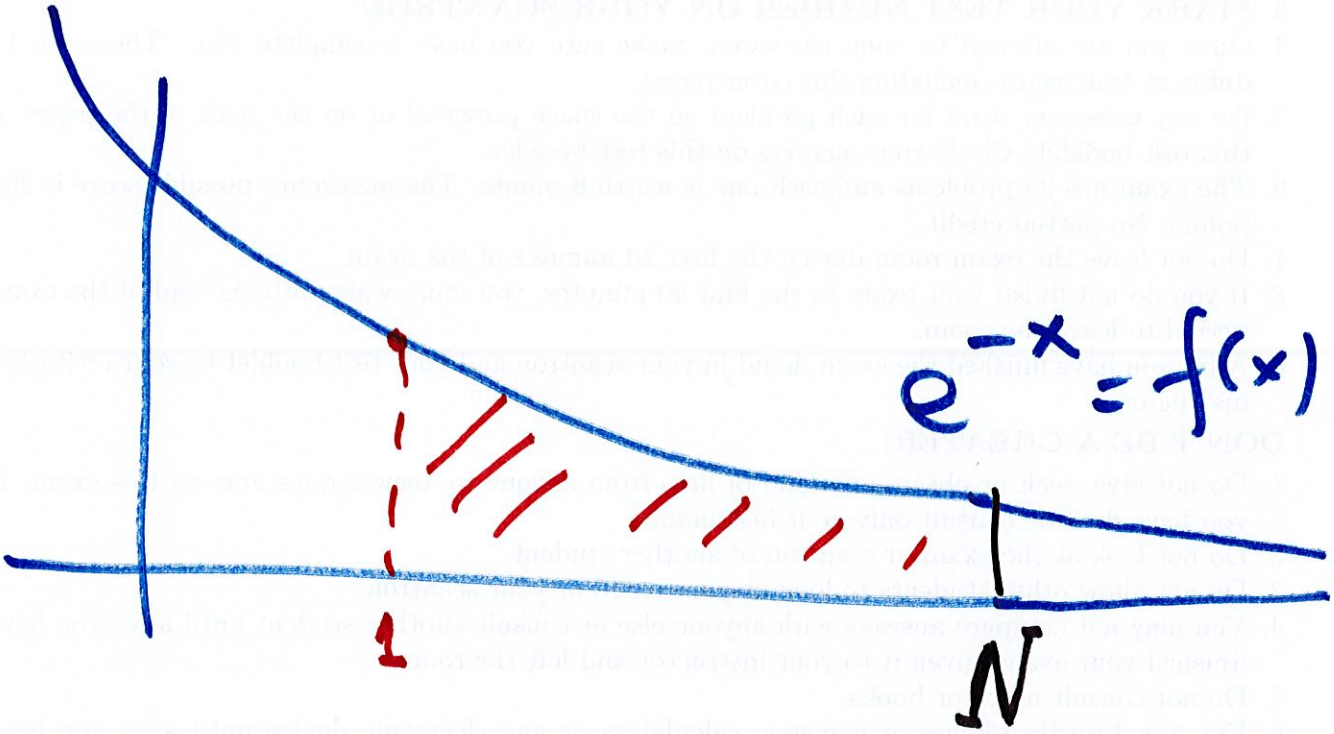
$$x+\sqrt{2} \geq \sqrt{x}$$

$$0 \leq \frac{1}{x+\sqrt{2}} \leq \frac{1}{\sqrt{x}}$$

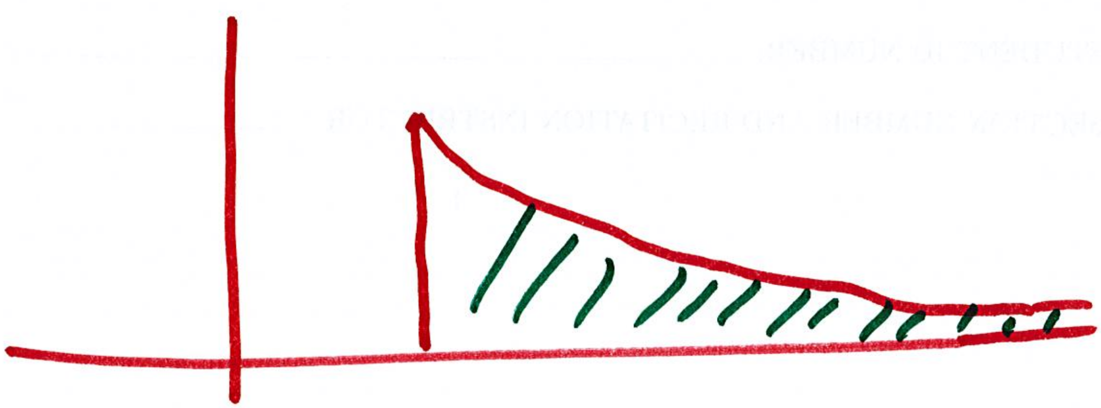
But we know that $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

We conclude that $\displaystyle\int_0^1 \frac{1}{x+\sqrt{x}}\,dx < \infty.$

Integrals on Infinite Intervals



$$\lim_{N \to \infty} \int_1^N e^{-x}\,dx = \int_1^\infty e^{-x}\,dx$$



$$\int_1^N e^{-x}\,dx = -e^{-x}\Big|_1^N$$

$$= e^{-1} - e^{-N}.$$

$$\lim_{N\to\infty} \int_1^N e^{-x}\,dx = e^{-1}.$$

$$\int_1^\infty x^{-p}\,dx = \lim_{N\to\infty} \int_1^N x^{-p}\,dx$$

$\underline{p=1}$

$$\int_1^N x^{-1}\,dx = \ln|x| \Big|_1^N$$

$$= \ln N.$$

$$\lim_{N\to\infty} \int_1^N x^{-1}\,dx = \int_1^\infty x^{-1}\,dx = \infty.$$

$\underline{p \neq 1}$

$$\int_1^N x^{-p}\,dx = \frac{x^{1-p}}{1-p} \Big|_1^N$$

$$= \frac{1}{1-p}\left[N^{1-p} - 1 \right]$$

when $1-p>0$

$$\lim_{N\to\infty}\int_1^N x^{-p}\,dx = \infty.$$

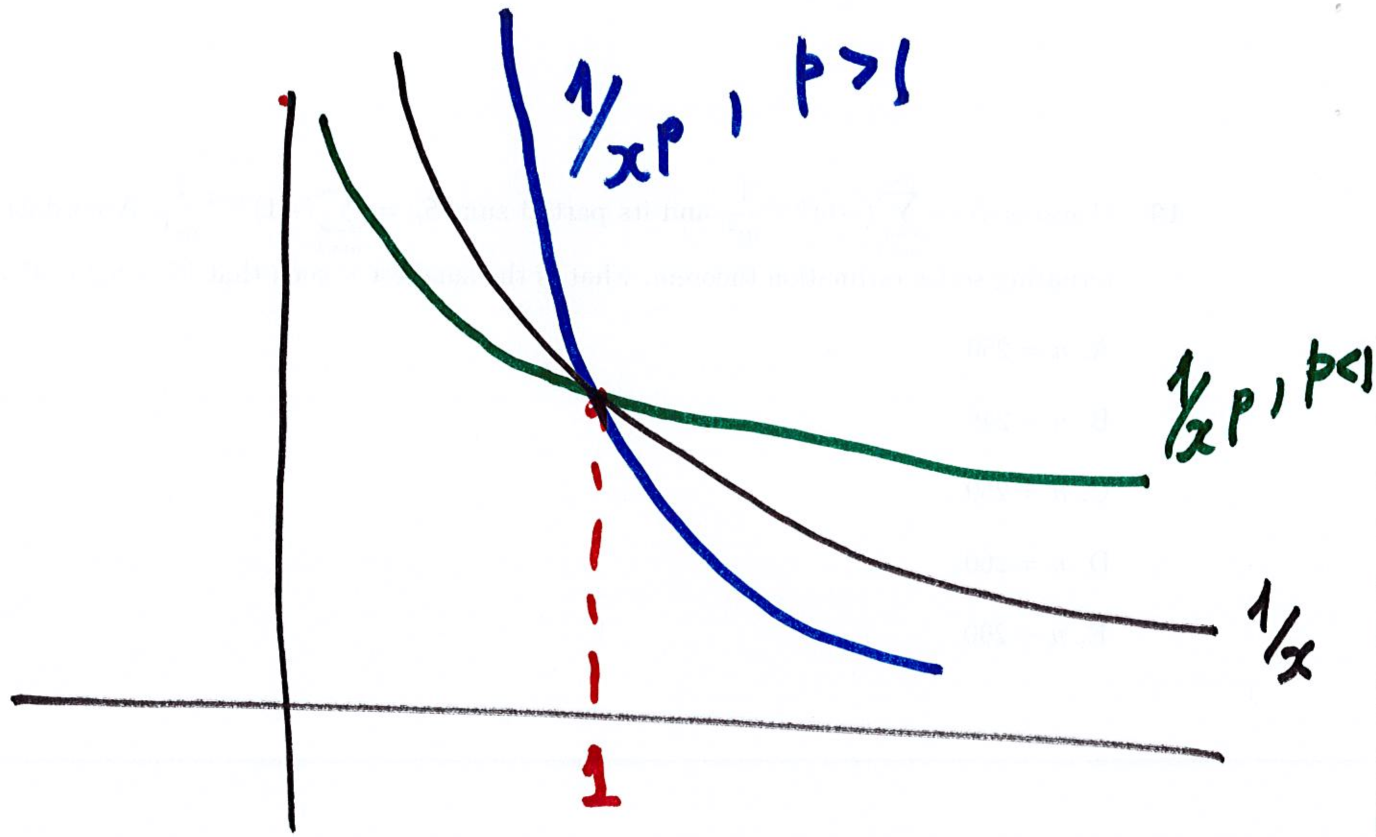
$$\int_1^\infty x^{-p}\,dx = \infty \text{ if } p<1$$

when $1-p<0$

$$\lim_{N\to\infty}\int_1^N x^{-p}\,dx = \frac{-1}{1-p}.$$

$\int_1^\infty x^{-p}\,dx$ converges

if $\underline{p>1}$.



$\int_0^1 1/x$ and $\int_1^\infty 1/x \, dx$ diverge

If $f(x) \geq g(x) \geq 0$ on $[1, \infty)$

If $\displaystyle\int_1^\infty f(x)\,dx < \infty$

then $\displaystyle\int_1^\infty g(x)\,dx < \infty$

If $\displaystyle\int_1^\infty g(x)\,dx = \infty$

then $\displaystyle\int_1^\infty f(x)\,dx = \infty$

Does

$$\int_1^\infty \frac{1}{x^2+5x+1}\,dx \text{ Converges?}$$

Converge or diverge?

for $x \geq 1$

$$x^2 + \underbrace{5x+1} > x^2$$

$$\frac{1}{x^2+5x+1} < \frac{1}{x^2}.$$

$$\int_1^\infty \frac{1}{x^2+5x+1}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx < \infty$$

So