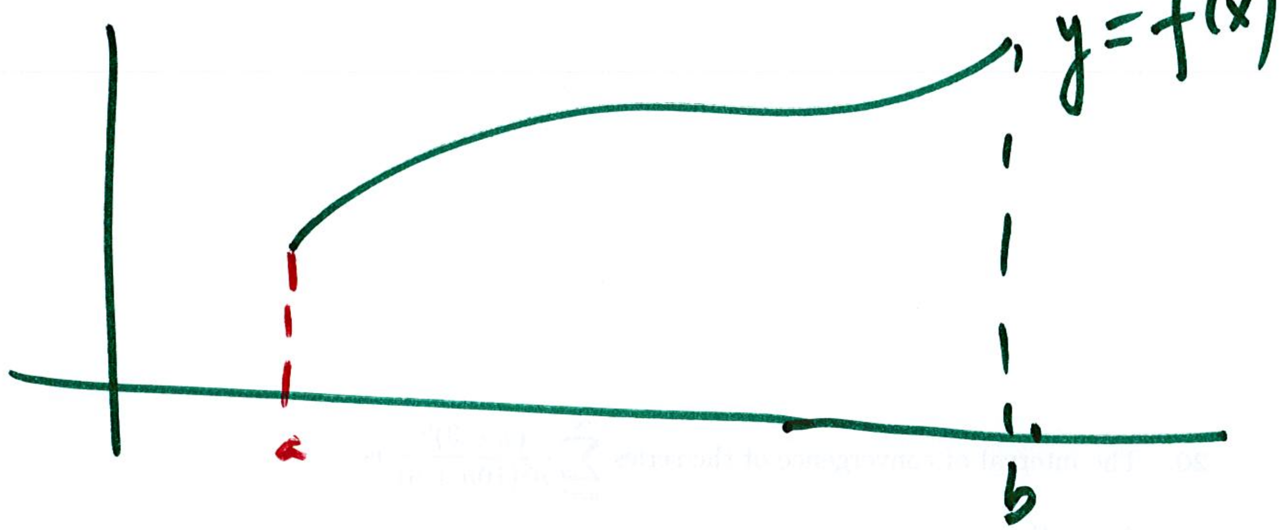
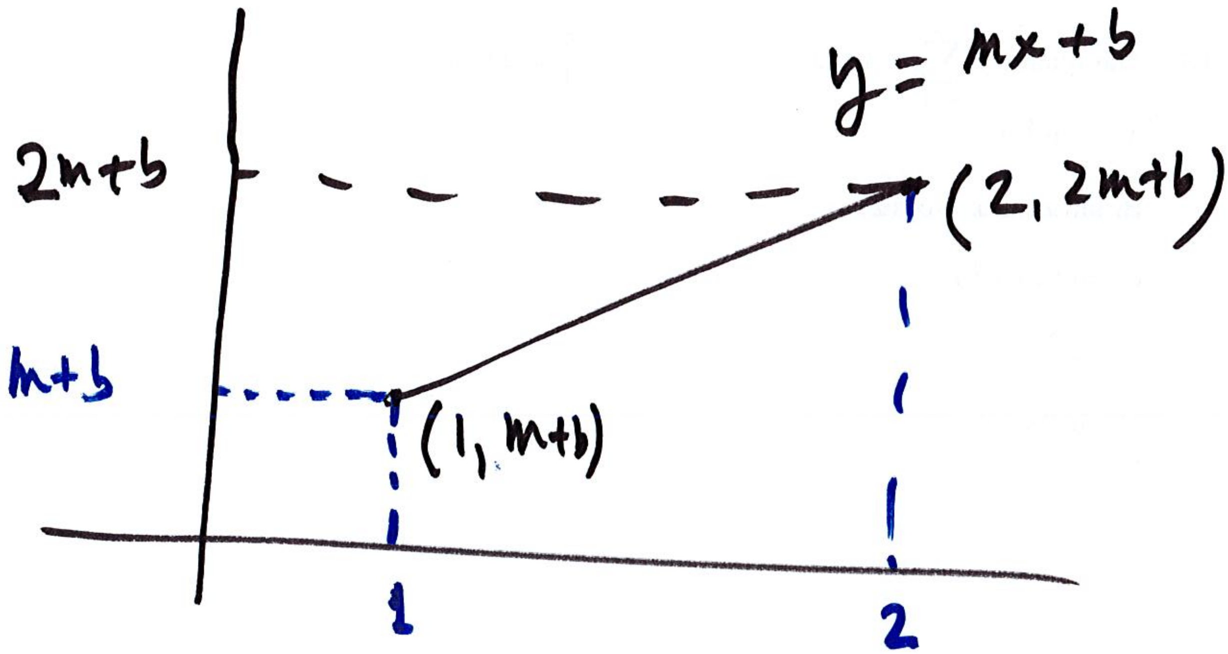
Lesson 17 Sections 8.1 and 8.2

Section 8.1 Arc Length



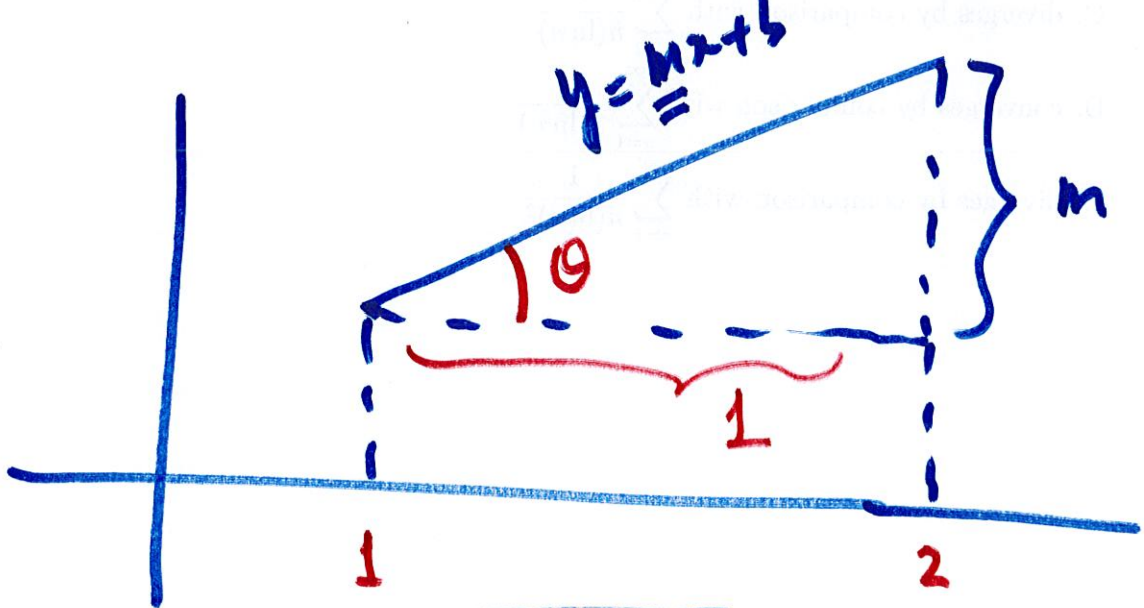
Goal: Compute the length of the curve described by

$$y = f(x)$$

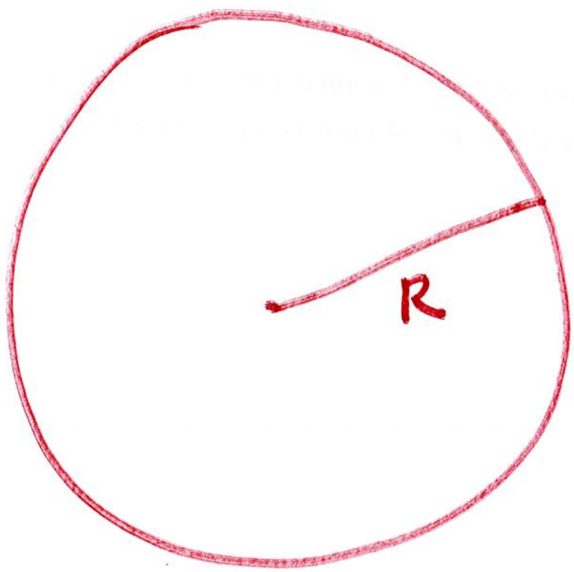


Length = Distance between the two points.

$\tan\theta = m$



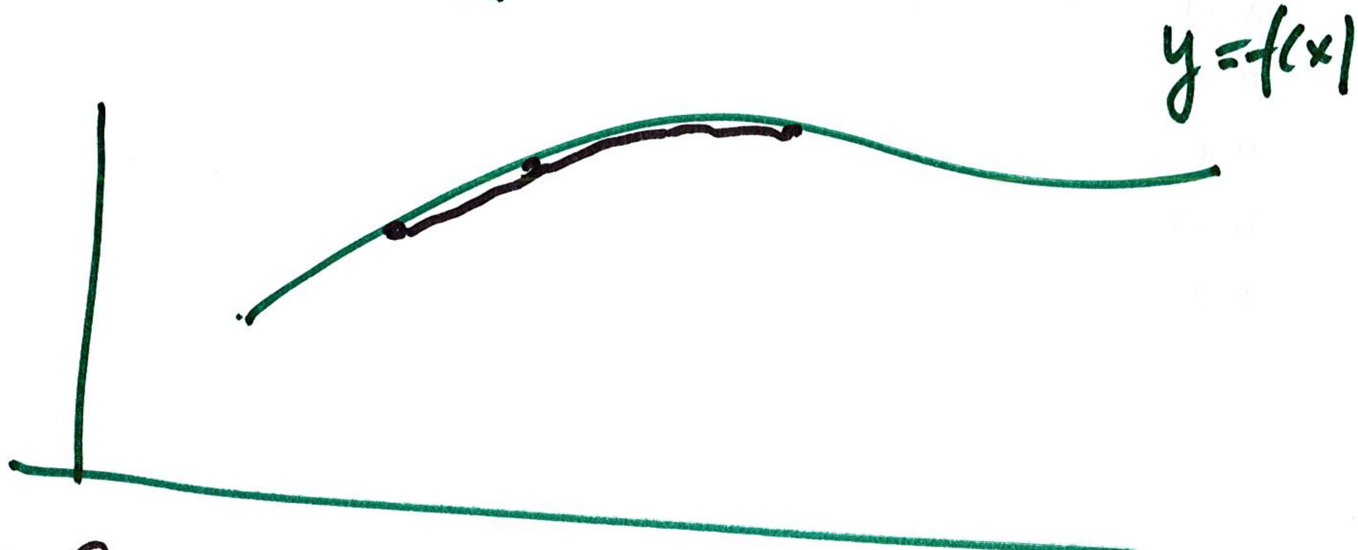
$L = \sqrt{1 + m^2}.$



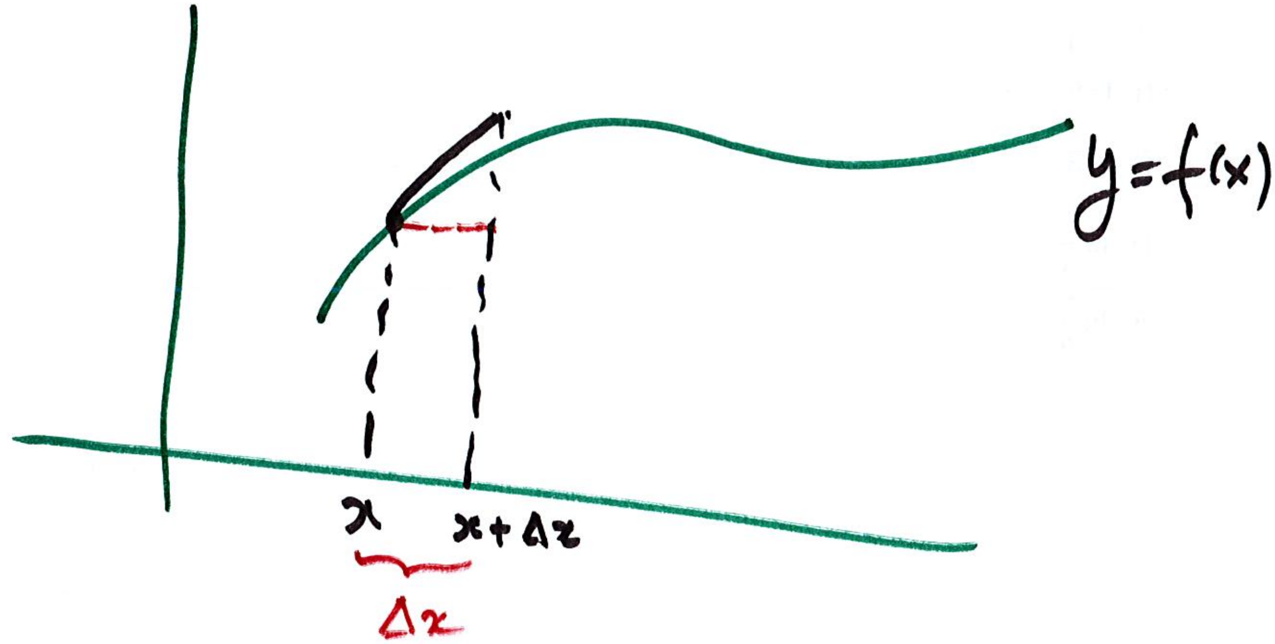
Length of the circumference

$= 2\pi R$.

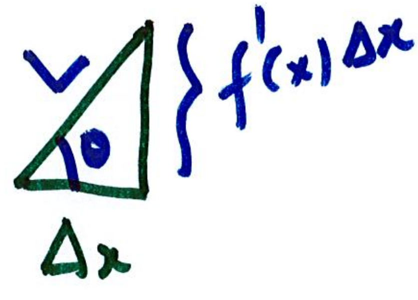
General Case



The principle is to approximate the curve by lines.



Approximate the graph of $f(x)$ by the line tangent to it at the point $(x, f(x))$

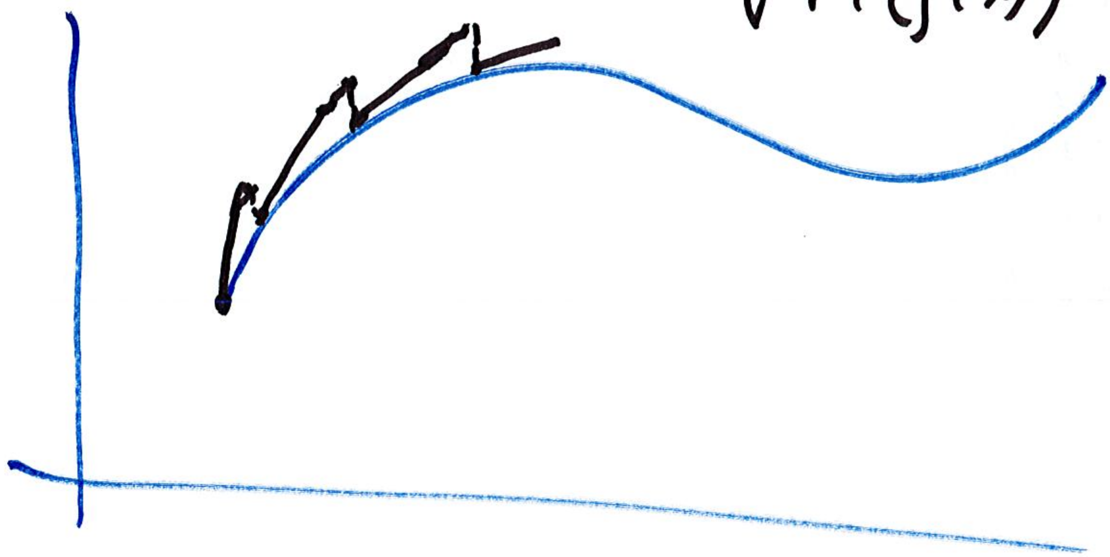


$\tan\theta = f'(x).$

$$L = \sqrt{(\Delta x)^2 + (f'(x))^2(\Delta x)^2} = \sqrt{1+(f'(x))^2}\,\Delta x$$

Eeach piece has length

$$\sqrt{1+(f'(x))^2}\,\Delta x$$



The Length of the curve

$$= \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Examples: $y = 1 + x^{3/2}$

for $1 \le x \le 2$.

$$L = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$f(x) = 1 + x^{3/2}$$

$$f'(x) = \tfrac{3}{2} x^{1/2}$$

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

$u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}\,dx$

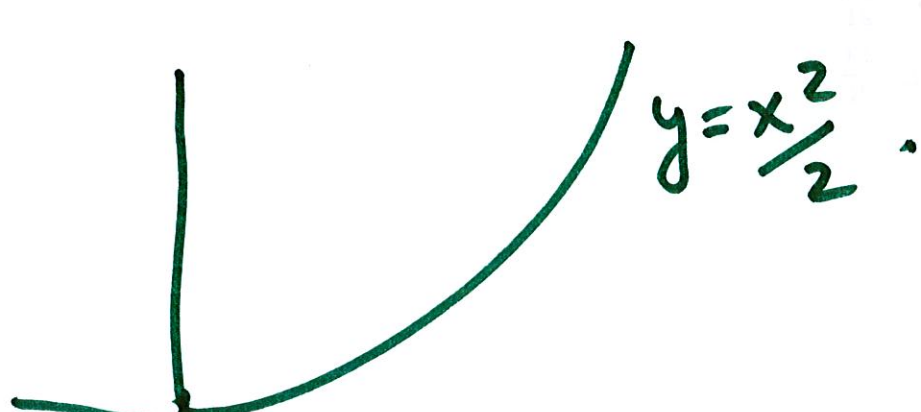
$dx = \frac{4}{9}\,du$

$$L = \frac{4}{9}\int_{13/4}^{11/2} \sqrt{u}\,.\ du$$

$$= \frac{4}{9}\cdot\frac{2}{3}\,u^{3/2}\Big|_{13/4}^{11/2}$$

$$= \frac{8}{27}\left(\left(\tfrac{11}{2}\right)^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right).$$

<u>Example:</u> $y = \frac{1}{2}x^2 \quad 0 \le x \le 1$



$$f(x) = x^2/2, \quad f'(x) = x$$

$$L = \int_0^1 \underbrace{\sqrt{1+x^2}}_{\sec\theta} \, \underbrace{dx}_{\sec^2\theta \, d\theta}.$$

$$x = \tan\theta, \quad \sqrt{1+x^2} = \sec\theta$$

$$L = \int_0^{\pi/4} \sec^3\theta \, d\theta =$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| \Bigg|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right).$$

$$y = \frac{x^3}{3} + \frac{1}{4x} \qquad 1 \leq x \leq 2.$$

$$f(x) = \frac{x^3}{3} + \frac{1}{4x} \ ; \ f'(x) = x^2 - \frac{1}{4x^2}$$

$$L = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$1 + (f'(x))^2 = 1 + \left(x^2 - \frac{1}{4x^2}\right)^2$$

$$= 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}$$

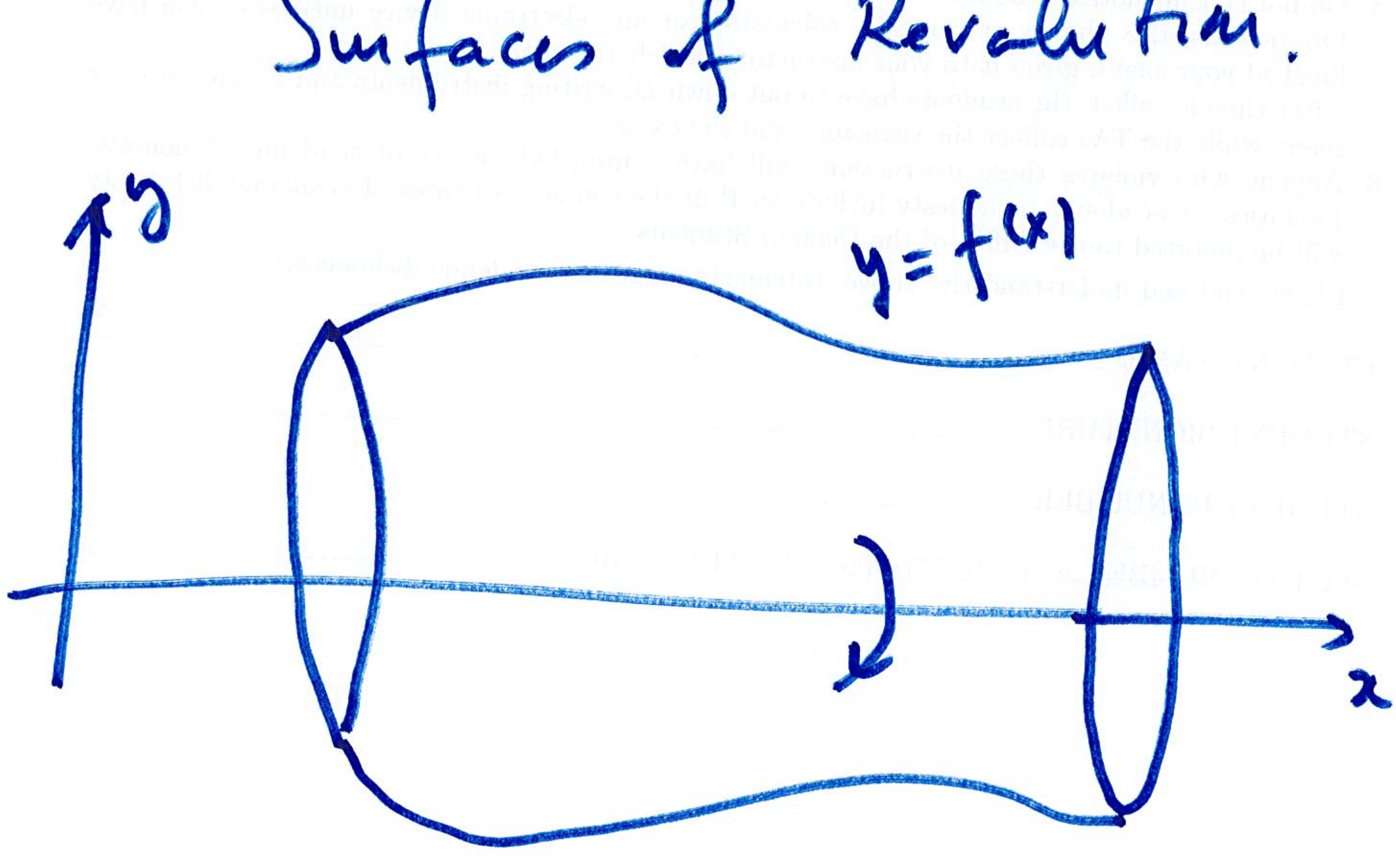
$$= x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2$$

$$\sqrt{1 + (f'(x))^2} = x^2 + \frac{1}{4x^2}.$$

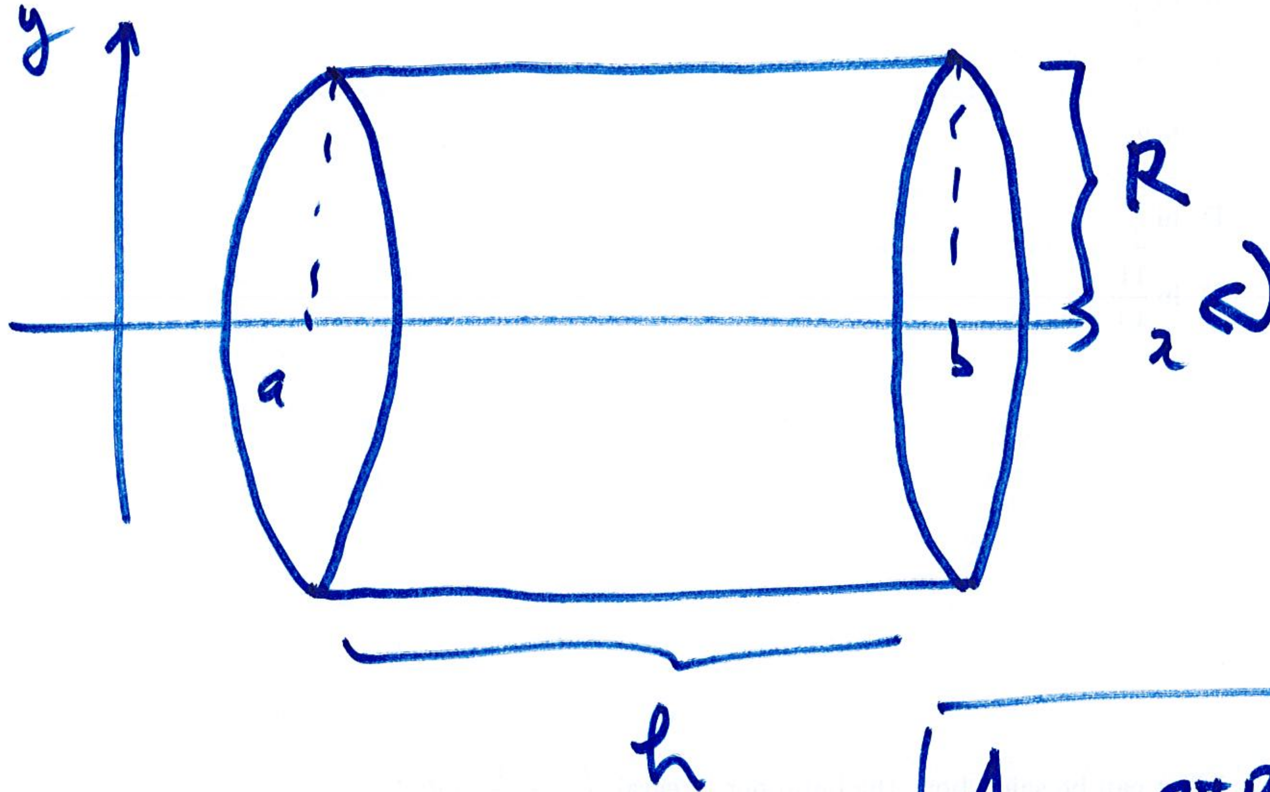
$$\int_1^2 \left(x^2 + \frac{1}{4x^2} \right) dx$$

$$= \frac{x^3}{3} - \frac{1}{4x} \Big|_1^2 = \ldots$$

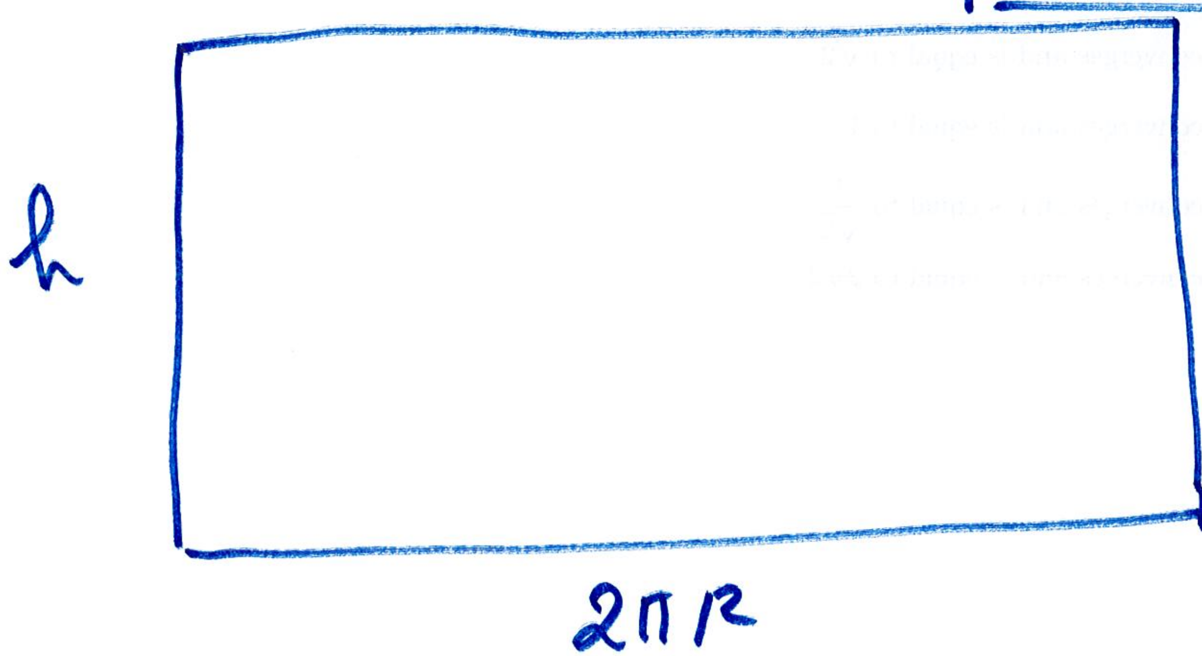
Section 8.2 Areas of Surfaces of Revolution.

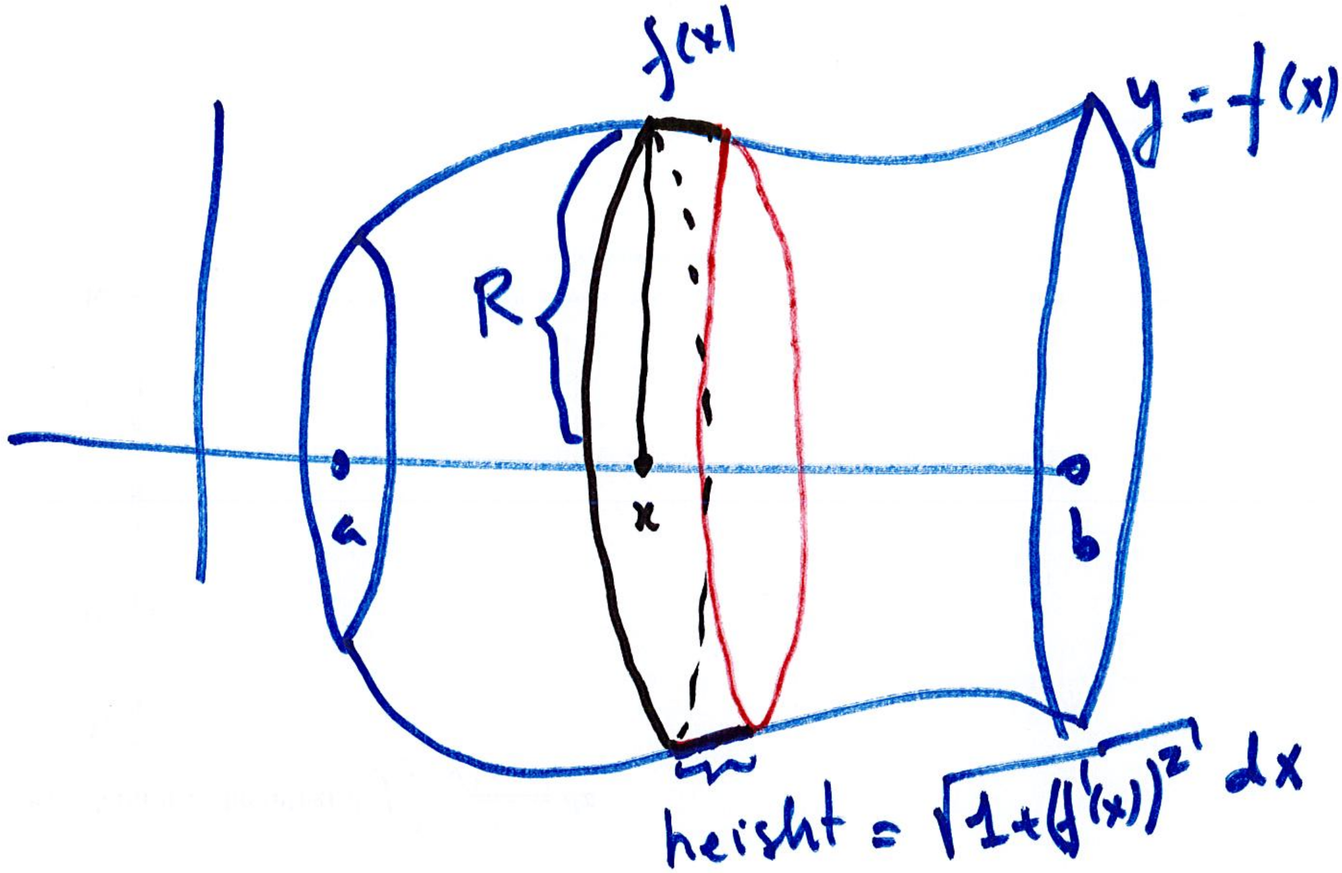


Cylinder



$$\boxed{Area = 2\pi R h}$$





Area of this little piece =

= 2π · Radius · height

Radius = f(x).

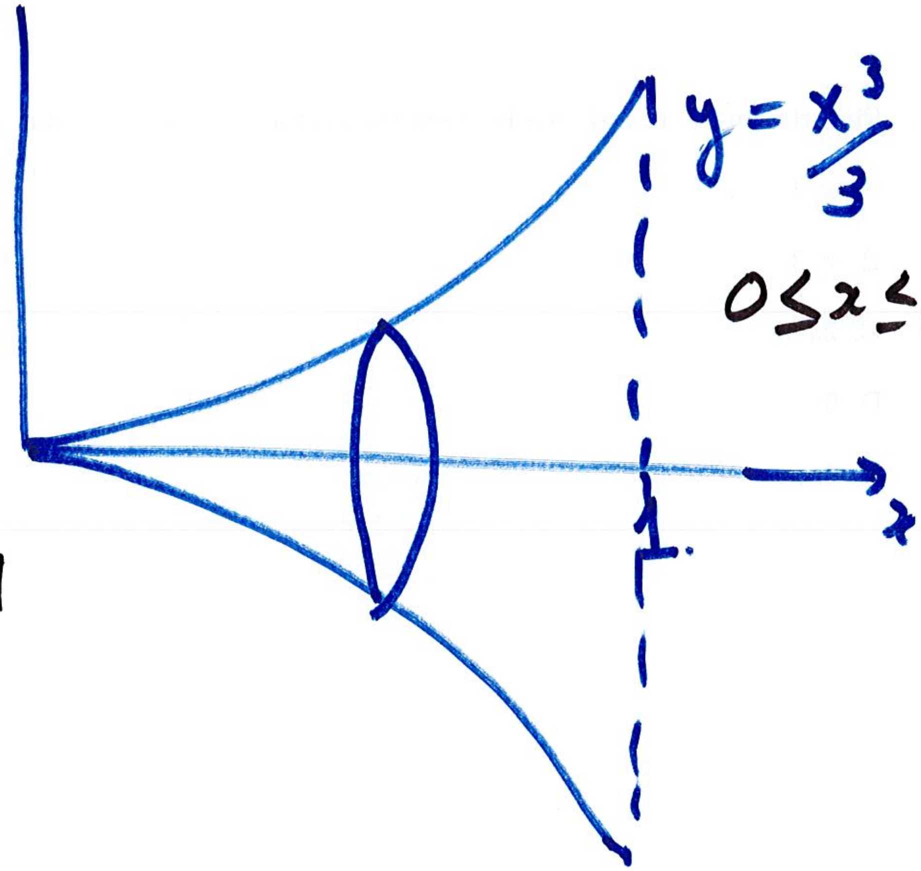
Height = $\sqrt{1+(f'(x))^2}\,dx$

$$\text{Area} = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

$f(x) = x^3/3$

$f'(x) = x^2$

$(f'(x))^2 = x^4$



$$A = \int_0^1 2\pi \frac{x^3}{3} \sqrt{1 + x^4}\, dx$$

$u = 1 + x^4$

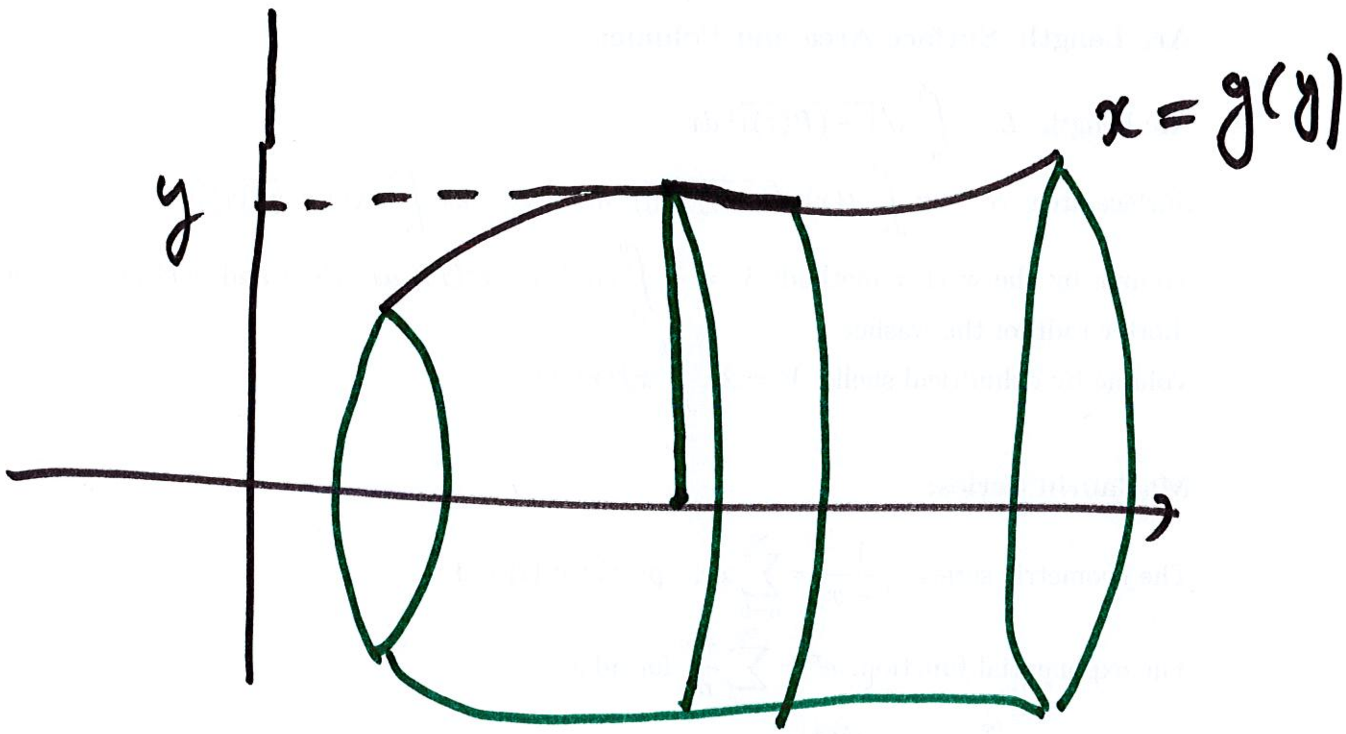
$du = 4x^3\, dx$

$$A = \frac{2\pi}{3} \int_1^2 u^{1/2} \frac{du}{4} = \frac{\pi}{6} \cdot \int_1^2 u^{1/2}\, du.$$

$$x = \frac{1}{3}\left(y^2+2\right)^{3/2} \quad 1 \le y \le 2$$

Rotate thus about the

x-axis.



Radius $= y$

$$\int_a^b 2\pi \cdot y \cdot \sqrt{1+(g'(y))^2}\, dy$$

In this case $g(y) = \frac{1}{3}(y^2+2)^{3/2}$

$$g'(y) = y\,(y^2+2)^{1/2}$$

$$1+(g'(y))^2 = 1 + y^2(y^2+2)$$

$$= 1+y^4+2y^2 = (1+y^2)^2$$

$$\sqrt{1+(g'(y))^2} = 1+y^2.$$

$$\text{Area} = 2\pi \int_1^2 y\,(1+y^2)\,dy = \ldots$$