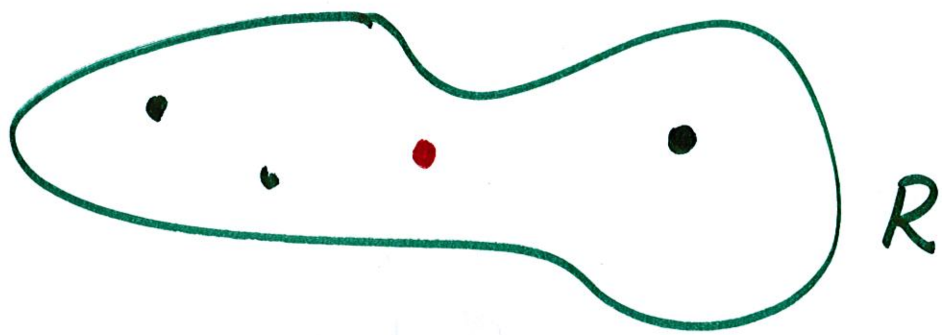
Lesson 18 Section 8.3

Center of Mass

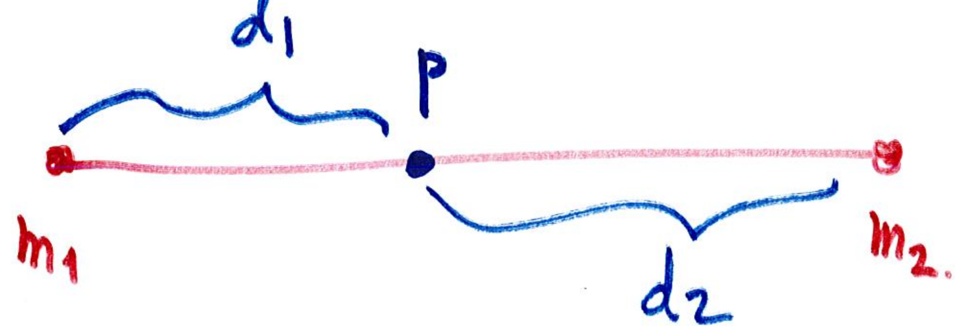
Goal:



Find the center of mass of a thin object (Laminas)

Assumption: The lamina is homogeneous = Constant density of mass.

1- Dimensional

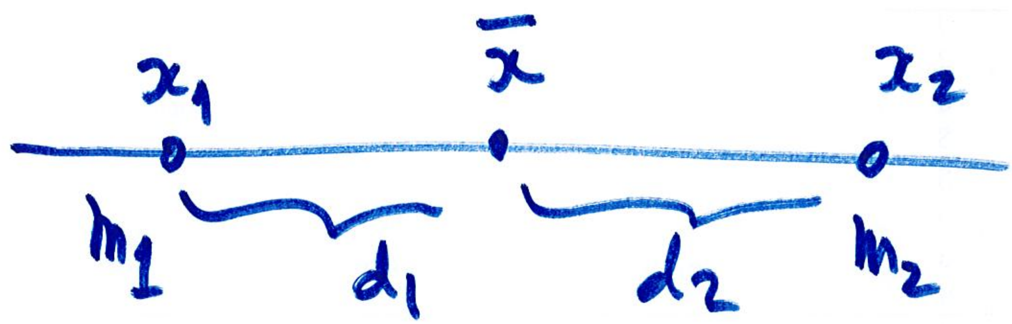


Moment: moment of m_1 with respect to p

$= m_1 d_1$

Same for m_2: $m_2 d_2$

Equilibrium: $m_1 d_1 = m_2 d_2$.



$\bar{x}$ = center of mass is the point such that

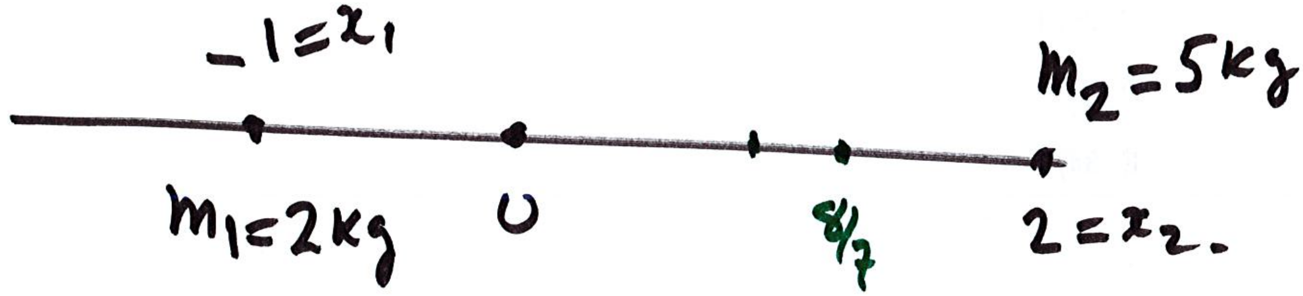
$$m_1(\underbrace{\bar{x}-x_1}_{d_1}) = m_2(\underbrace{x_2-\bar{x}}_{d_2})$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

Example:



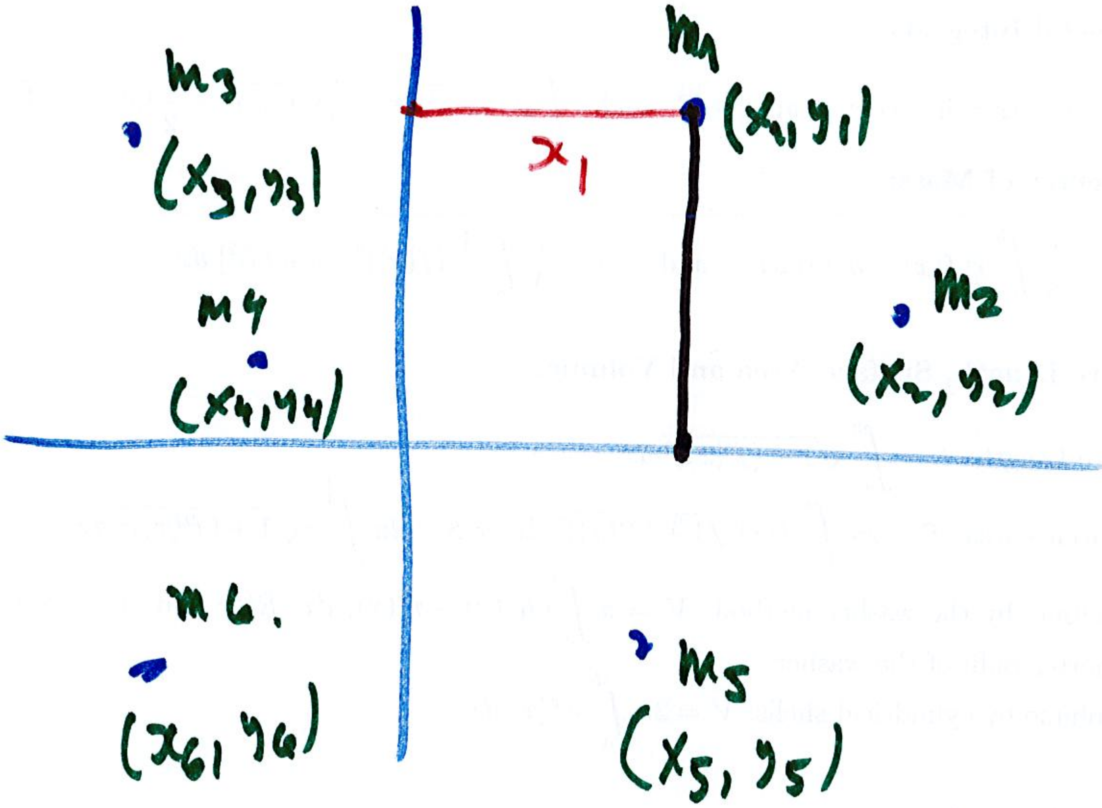
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{-2+10}{7}$$

$$\bar{x} = 8/7$$

Several objects: $m_1, m_2, \ldots$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n}.$$

Two Dimensional objects



Moment with respect to the

x-axis. -

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 + \dots + m_n y_n$$

Moment with respect to the y-axis

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n.$$

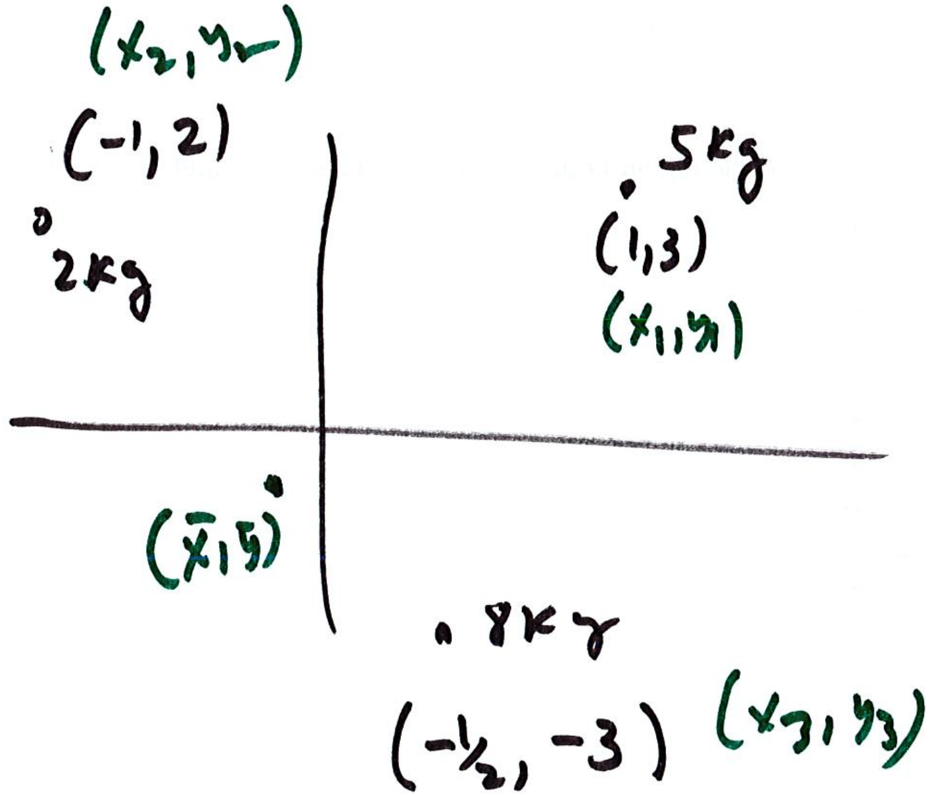
The center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{M_y}{M}$$

M_y = moment with respect to the y-axis

M = total mass.

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n} = \frac{M_x}{M}.$$

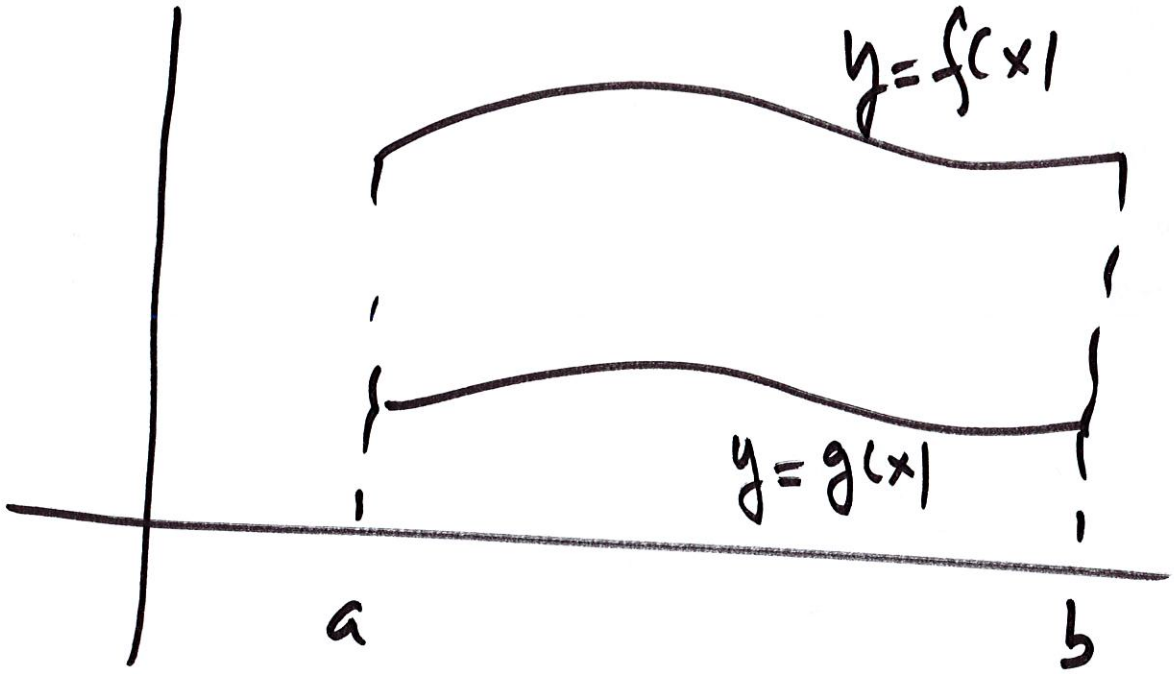


$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}.$$

$$\bar{x} = \frac{5 + (-2) + (-4)}{15} = \frac{-1}{15}.$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{15 + 4 - 24}{15}$$

$$= -\frac{1}{3}$$



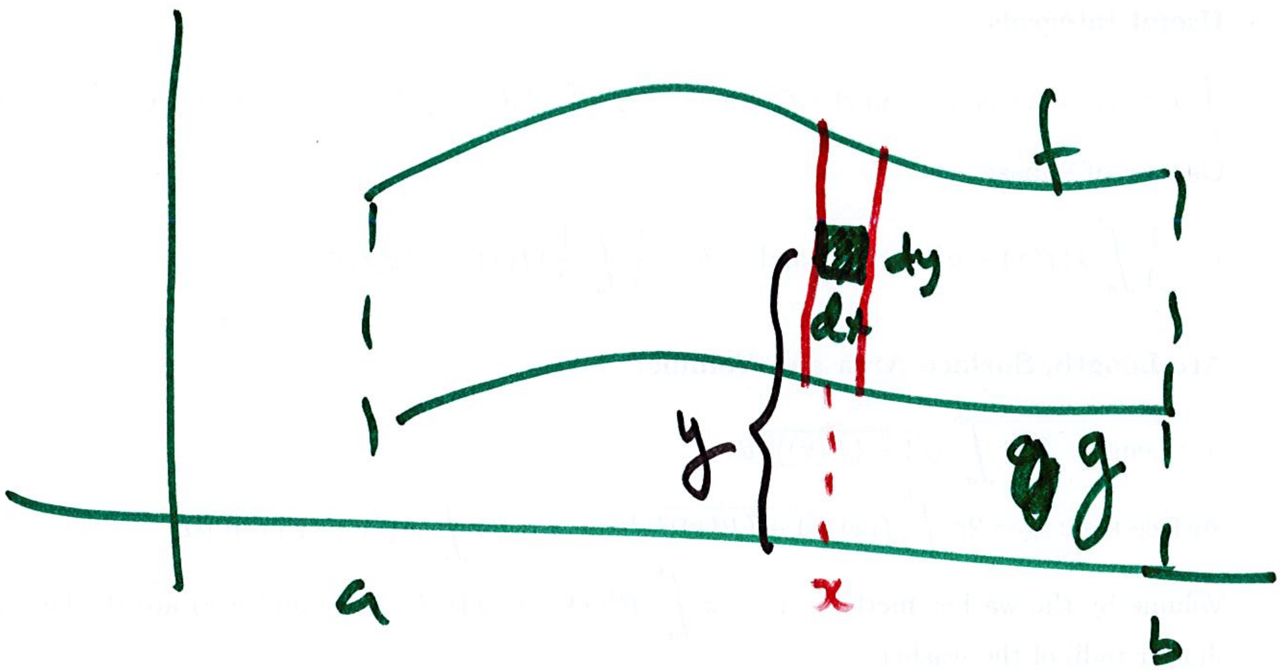
Mass = $\underbrace{\text{density}}_{\rho}$ × Area = ρA.

$$A = \int_a^b (f(x) - g(x))\,dx$$

Density is Constant

M_x

$$\bar{x} = \frac{M_y}{M} \; ; \; \bar{y} = \frac{M_x}{M}.$$



mass $= \rho \, dx \, dy.$, ρ = density of mass

$$dM_x = y \rho \, dx \, dy$$

M_x of the strip $= \left(\int_{g(x)}^{f(x)} y \rho \, dy \right) dx.$

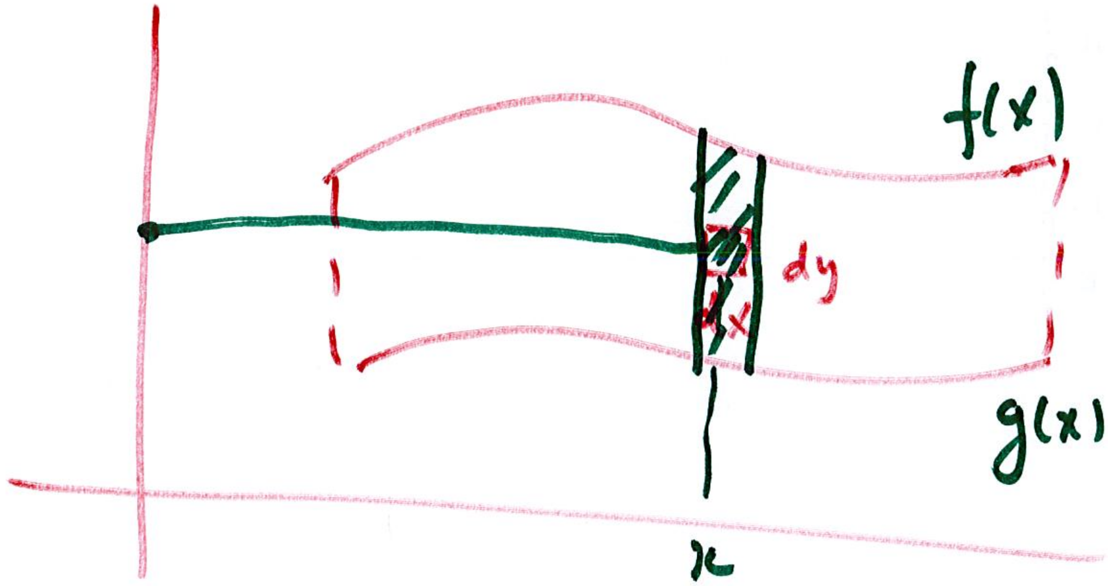
$$= \left(\frac{\rho}{2} y^2 \Big|_{g(x)}^{f(x)} \right) dx = \frac{\rho}{2} \left(f^2(x) - g^2(x) \right) dx$$

Moment M_x for the whole region

$$M_x = \frac{\rho}{2} \int_a^b \left[f^2(x) - g^2(x) \right] dx.$$

$$\bar{y} = \frac{M_x}{M}, \quad M = \rho \int_a^b \left(f(x) - g(x) \right) dx.$$

M_y



$$dM_y = x \, \rho \, dx \, dy$$

Moment of the strip $= \left[\int_{g(x)}^{f(x)} x \, \rho \, dy \right] dx$

$$= \rho x \left(f(x) - g(x) \right) dx$$

$$\boxed{M_y = \rho \int_a^b x \left(f(x) - g(x) \right) dx}$$

$$\bar{x} = \frac{M_y}{M}$$

$$M_y = \rho \int_a^b x \left(f(x) - g(x) \right) dx.$$

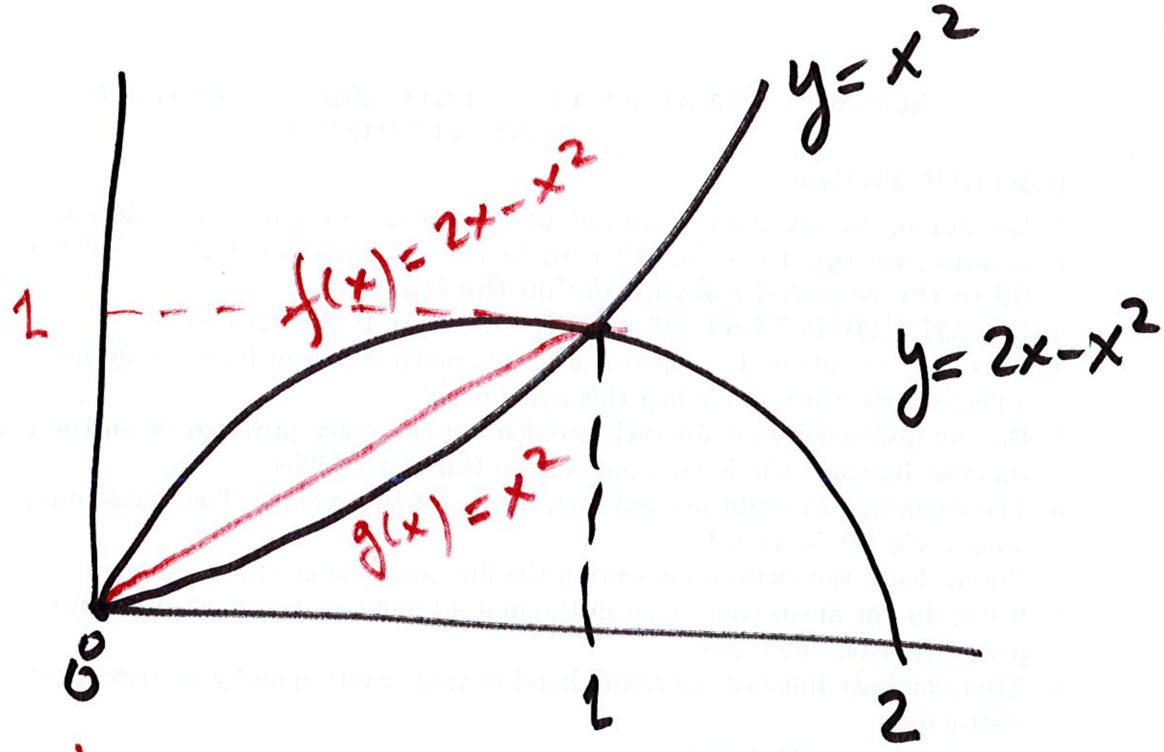
$$M = \rho \int_c^b \left(f(x) - g(x) \right) dx.$$

Example: Find the center of mass of the region bounded by $y = x^2$ and $y = 2x - x^2$.

density of mass $= \rho$ constant.

(Center of mass = Centroid)

Step 1: Identify who $f(x)$ and $g(x)$ are.



(1) $$M = \int_0^1 \left(f(x) - g(x)\right) dx$$

$$2x - x^2 = x^2$$

$$2x = 2x^2$$

$$x = 0$$

$$2 = 2x$$

$$x = 1$$

$$= \int_0^1 \left[2x - x^2 - x^2\right] dx$$

$$= 2\int_0^1 (x - x^2)\, dx = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

$$M_y = \int_0^1 x\,(\underbrace{f(x) - g(x)}_{2x-2x^2})\,dx$$

$$= \int_0^1 x\,(2x - 2x^2)\,dx$$

$$= 2\int_0^1 (x^2 - x^3)\,dx$$

$$= 2\left[\tfrac{1}{3} - \tfrac{1}{4}\right] = 2 \cdot \frac{1}{12} = \tfrac{1}{6}.$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/6}{1/3} = \tfrac{1}{2}.$$

$$M_x = \frac{1}{2}\int_0^1 \left[(f(x))^2 - (g(x))^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[(2x-x^2)^2 - (x^2)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[4x^2 - 4x^3 + x^4 - x^4\right] dx$$

$$= \frac{1}{2}\int_0^1 (4x^2 - 4x^3)\, dx$$

$$= 2\int_0^1 (x^2 - x^3)\, dx$$

$$= 2\left(1/3 - 1/4\right) = 1/6 .$$