Lesson 20 11.1

Sequences (and Series)

Compute Limits of Some Sequences.

Last time

$$\lim_{n\to\infty} \frac{n^4+4n^3+5n+1}{n^4+8n^2+3n} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^4}\left(1+4\frac{n^3}{n^4}+5\frac{n}{n^4}+\frac{1}{n^4}\right)}{\cancel{n^4}\left(1+8\frac{n^2}{n^4}+3\frac{n}{n^4}\right)}$$

$$= \lim_{n\to\infty} \frac{1+4/n+5/n^3+1/n^4}{1+8/n^2+3/n^3} = 1$$

$$\lim_{n \to \infty} \frac{P(n)}{Q(n)} =$$

If $P(n)$ and $Q(n)$ have the same degree.

$$P(n) = a_0 n^k + a_1 n^{k-1} + \dots$$

$$Q(n) = b_0 n^k + b_1 n^{k-1} + \dots$$

$$\lim_{n \to \infty} \frac{P(n)}{Q(n)} = \frac{a_0}{b_0}$$

If the degree of

$Q(n)$ is higher than

the degree of $P(n)$

$$\cancel{\lim_{n\to\infty} \frac{Q(n)}{P(n)}} =$$

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = 0$$

If $P(n)$ is of higher degree

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \infty.$$

Determine if

$$\boxed{a_n = n e^{-n}}$$

is increasing or decreasing

or neither.

$f(x) = x e^{-x}$, $a_n = f(n)$.

If $f(x)$ is decreasing or increasing, so is the sequence.

$f(x) = x e^{-x}$

$f'(x) = e^{-x} - x e^{-x}$

$= \underbrace{e^{-x}}_{>0} (1-x)$

$f(x)$ is decreasing if $1-x < 0$

$\boxed{x > 1.}$

The sequence $a_n = f(n)$

$n = 1, 2, 3, \ldots$

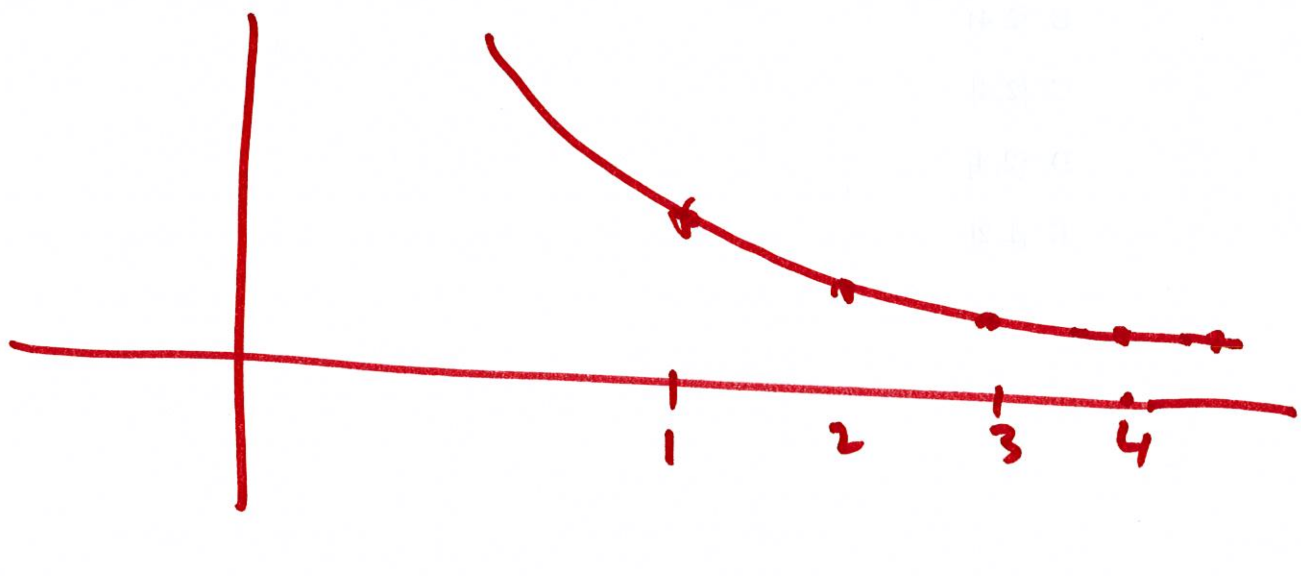
So a_n is decreasing.

Find $\lim_{n \to \infty} n e^{-n}$

Consider $f(x) = x e^{-x}$

$$n e^{-n} = f(n)$$

Compute $\lim_{x \to \infty} f(x)$



$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x^{\to\infty}}{e^{x}_{\to\infty}}$$

$$= \lim_{x\to\infty} \frac{1}{e^{x}} = 0$$

$$\lim_{x\to\infty} x e^{-x} = 0$$

$$\lim_{n\to\infty} n e^{-n} = 0.$$

$$\lim_{n\to\infty} n \sin(1/n)$$

Consider $f(x) = x \sin(1/x)$

$$\lim_{x\to\infty} x \sin(1/x) =$$

$$\lim_{x\to\infty} \frac{\overbrace{\sin(1/x)}^{\to 0}}{\underbrace{1/x}_{\to 0}}$$

$$= \lim_{x\to\infty} \frac{\cos(1/x)\cdot \cancel{-1/x^2}}{\cancel{-1/x^2}}$$

$$= \lim_{x\to\infty} \frac{\cos(1/x)}{1} = 1.$$

$a_n = r^n$

r is a <u>real number</u>

$$\lim_{n \to \infty} r^n = ?$$

<u>Case 1</u>: If $|r| < 1$.

$$\lim_{n \to \infty} r^n =$$

<u>Example:</u> $r = 1/2$.

$$a_n = \left(\tfrac{1}{2}\right)^n$$

$$a_1 = \tfrac{1}{2}, \quad a_2 = \tfrac{1}{4}, \quad a_3 = \tfrac{1}{8}$$

$\ldots$

$$a_n = r^n \quad , 0 < r < 1$$

$$a_1 = r, \quad a_2 = r^2$$

$$0 < r < 1$$

$$0 < r^2 < r$$

$$0 < r^3 < r^2$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (r)^n = 0 \quad \text{if } 0 < r < 1$$

$r = -1/2$

$a_n = r^n$

$a_1 = -1/2, \quad a_2 = 1/4, \quad a_3 = -\frac{1}{8}$

$a_4 = 1/16, \ldots$

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } -1 < r < 0.$$

$$\boxed{\lim_{n\to\infty} r^n = 0 \quad \text{if } |r| < 1}$$

If $r > 1$

$$\lim_{n \to \infty} r^n = \infty.$$

If $r < -1$

n is odd

$$\lim_{n \to \infty} r^n = -\infty$$

n is even

$$\lim_{n \to \infty} r^n = \infty.$$

If $r < -1$

$\lim_{n \to \infty} r^n$ does not exist

$r = -1$ $\qquad (-1)^n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

$\lim_{n \to \infty} (-1)^n$ does not exist.

Series

We start with a sequence

$$a_1, a_2, a_3, a_4, \dots$$

Define another sequence

$$\boxed{S_N = a_1 + a_2 + a_3 + a_4 + \dots + a_N.}$$

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_N = \sum_{n=1}^{N} a_n$$

$$= a_1 + a_2 + a_3 + \cdots + a_N$$

S_N is called the sequence of partial sums of

a_n

$$S = \lim_{N \to \infty} \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n$$

is called the series.

Does it converge?

Example:

$$a_n = r^n \qquad n = 1, 2, 3, \dots$$

$$a_0 = 1.$$

$$1, r, r^2, r^3, r^4, \dots$$

Partial Sums

$$S_N = 1 + r + r^2 + \dots + r^N$$

$$= \sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

Question For what values of r does this converge?

$$S_N = \underline{1} + \cancel{r} + \cancel{r^2} + \dots + r^{N-1} + \cancel{r^N}$$

multiply this by r.

$$r S_N = \cancel{r} + \cancel{r^2} + r^3 + \dots + \cancel{r^N} + r^{N+1}$$

$$\boxed{S_N - r S_N = 1 - r^{N+1}}$$

$$S_N (1-r) = 1 - r^{N+1}$$

$$\boxed{S_N = \frac{1 - r^{N+1}}{1-r}}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r}.$$

$$\lim_{N\to\infty} r^{N+1} = 0 \quad \text{if } |r|<1$$

diverges when $|r|>1$

or $\underline{r=-1}$

Diverges if $|r|\geq 1$.

$$\boxed{\lim_{N\to\infty} S_N = \frac{1}{1-r}}$$

provided $|r|<1$.

Compute $\sum_{n=0}^{\infty} \frac{8^{n+1}}{10^n} = \underline{40}$

$$= \sum_{n=0}^{\infty} \underline{8} \cdot \frac{8^n}{10^n}$$

$$= 8 \sum_{n=0}^{\infty} \underbrace{\left(\frac{8}{10}\right)}_{= r}^n \qquad S = \sum_{n=0}^{\infty} r^n$$

$$\boxed{r = 8/10 = 4/5.}$$

$$\sum_{n=0}^{\infty} \left(\frac{8}{10}\right)^n = \frac{1}{1 - 4/5} = 5$$