Lesson 21 Sections 11.2 and 11.3

Sequences and Series

We start with a sequence

a_n, $a_1, a_2, a_3, \ldots$

Form another sequence

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_N = a_1 + a_2 + a_3 + \cdots + a_N.$

S_N = Sequence of partial Sums

<u>Question:</u> Does S_N converge

Does $\lim_{N\to} S_N = \sum_{n=1}^{\infty} a_n$

exist?

<u>Example:</u> • Pick r in $(-1,1)$

$a_0 = 1$

$a_1 = r$

$a_2 = r^2$

$a_3 = r^3, \ldots$

$$S_N = 1 + r + r^2 + \dots + r^N$$

$$= \sum_{n=0}^{N} r^n$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} r^n$$

$$= \frac{1}{1-r}.$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$ if r is in $(-1, 1)$

If r is not in $(-1,1)$ the series diverges.

Find the sum

$$\sum_{n=0}^{\infty} (0.2)^n$$

$r = 0.2$

$$\boxed{\sum_{n=0}^{\infty} (0.2)^n = \frac{1}{1-0.2} = \frac{1}{0.8}}$$

Question:

$$r^k + r^{k+1} + r^{k+2} + \cdots = ?$$

$$\boxed{1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Notice that

$$r^k + r^{k+1} + r^{k+2} + \cdots =$$

$$r^k \Big(\underbrace{1 + r + r^2 + \cdots}_{1/(1-r)} \Big) = \frac{r^k}{1-r}.$$

Example: Compute

$$\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \dots$$

$$= \left(\frac{1}{3}\right)^4 \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots\right]$$

$$= \frac{\left(\frac{1}{3}\right)^4}{1 - 1/3}.$$

$$\boxed{\sum_{n=10}^{\infty} (0.8)^n = \frac{(0.8)^{10}}{1-0.8}}$$

In general we cannot compute the sum of a series.

We can only say if

$$\sum_{n=1}^{\infty} a_n$$

converges or not.

$$\sum_{n=1}^{\infty} \frac{n+5}{n+8}$$

$$= \frac{6}{9} + \frac{7}{10} + \frac{8}{11} + \cdots$$

Test For Divergence

$$S_N = \sum_{n=1}^{N} a_n$$

$$= a_1 + a_2 + \cdots + a_N$$

If $\lim_{N \to \infty} S_N = S$, the series converges.

$$S_{N+1} = a_1 + a_2 + \dots + a_N + a_{N+1}$$

$$S_{N+1} - S_N = a_{N+1}$$

$$\lim_{N\to\infty} \left(\underline{\underline{S_{N+1}}} - \underline{\underline{S_N}} \right) = S - S = 0$$

$$\lim_{N\to\infty} a_{N+1} = \lim_{N\to\infty} a_N = 0$$

$$\boxed{\underline{\text{Conclusion:}}\ \text{If } \lim_{N\to\infty} S_N = S \text{ then } \lim_{N\to\infty} a_N = 0}$$

The Divergence Test

$$\sum_{n=1}^{\infty} a_n$$

If $\lim_{n\to\infty} a_n$ either does not exist

~~or~~ or if it exists is not equal to zero.

The series $\sum_{n=1}^{\infty} a_n$

diverges.

Does $\sum_{n=1}^{\infty} \frac{n+5}{n+8}$ converge?

a_n

$$a_n = \frac{n+5}{n+8}$$

$$\boxed{\lim_{n\to\infty} a_n = 1}$$

$\sum_{n=1}^{\infty} \frac{n+5}{n+8}$ does not converge.

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

$$a_n = \cos(1/n)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(1/n)$$

$$= 1$$

$\sum_{n=1}^{\infty} \cos(1/n)$ diverges.

However, if

$$\lim_{n \to \infty} a_n = 0$$

the series

$$\sum_{n=1}^{\infty} a_n$$

may converge or may diverge.

We will see today

that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges

But

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges.

In the first case $a_n = 1/n$

$$\lim_{n\to\infty} a_n = 0$$

In the second case $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} a_n = 0.$$

The Integral test

$$\boxed{\sum_{n=1}^{\infty} a_n}$$ where

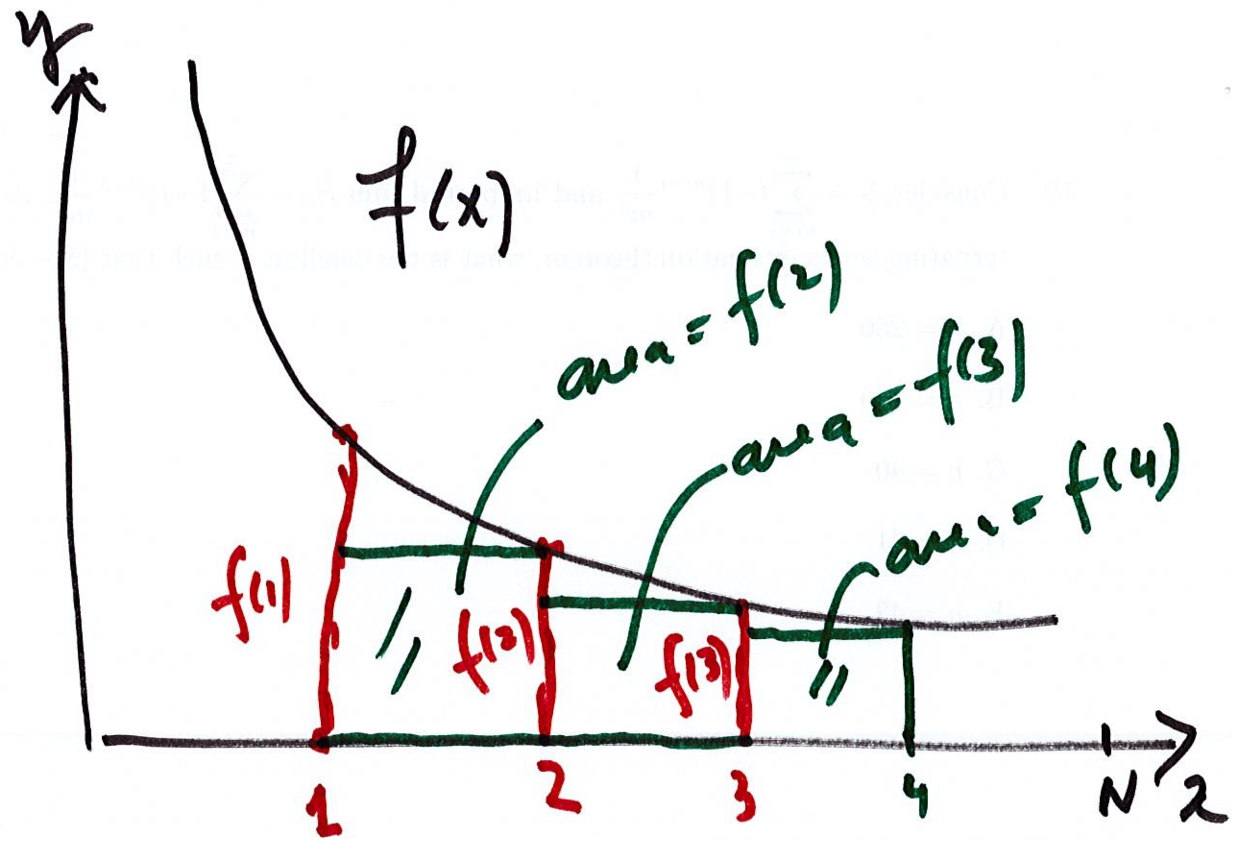
$$a_n = f(n)$$

where f is a function

Example: $f(x) = \frac{1}{x}$

$$f(n) = 1/n$$

$$f(x) = \frac{1}{x^2} \; ; \; f(n) = \frac{1}{n^2}.$$



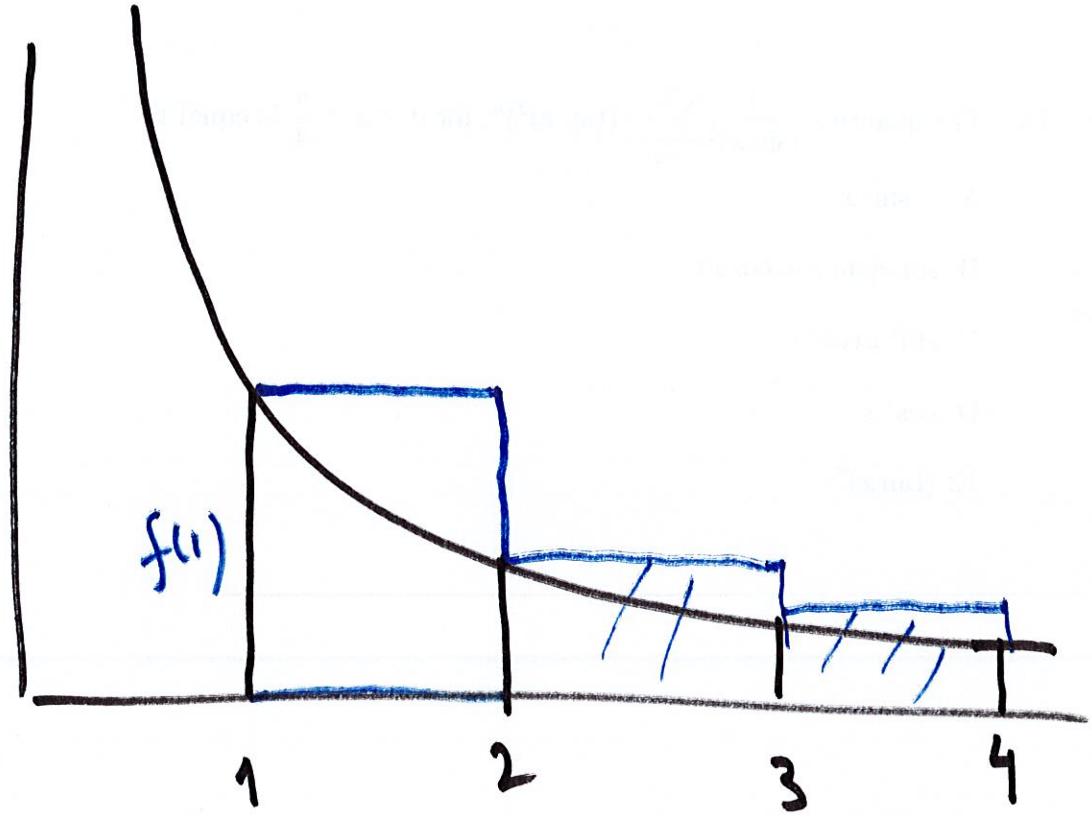
$$f(1) + f(2) + f(3) + \cdots$$

Series $\sum_{n=1}^{\infty} a_n \qquad a_n = f(n).$

$$f(2) + f(3) + \cdots + f(N) \leq \int_1^N f(x)\,dx$$

$$\lim_{N \to \infty} \sum_{n=2}^{N} f(n) =$$

$$= \underbrace{\sum_{n=2}^{\infty} f(n)} \le \int_{1}^{\infty} f(x)\,dx$$

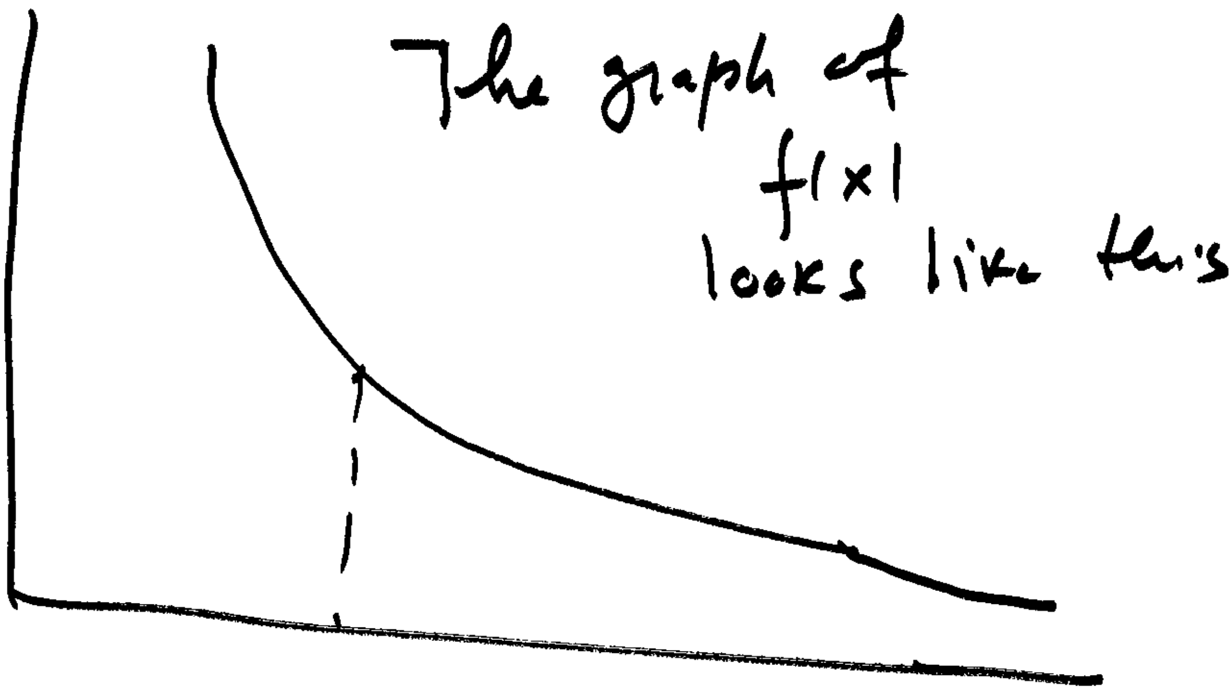


$$f(1)+f(2)+f(3)+\dots+f(N) \geq \int_1^N f(x)\,dx$$

$$\sum_{n=1}^{\cancel{\infty}} f(n) \geq \int_1^N f(x)\,dx.$$

$$\lim_{N\to\infty}\sum_{n=1}^{N} f(n) = \underbrace{\sum_{n=1}^{\infty} f(n)} \geq \int_1^{\infty} f(x)\,dx.$$

Conclusion:



If

1) $f(x) > 0$

2) $f(x)$ is decreasing for $x \geq 1$

3) $\lim_{x \to \infty} f(x) = 0$

Then the series

$$\sum_{n=1}^{\infty} f(n)$$ ~~an~~ and

$$\int_1^{\infty} f(x)\,dx$$

Converge or diverge ~~to the~~ Together

If $\displaystyle\sum_{n=1}^{\infty} f(n)$ Converges $\displaystyle\int_1^{\infty} f(x)\,dx$ Converges

If $\displaystyle\cancel{\sum_{n}} \int_1^{\infty} f(x)\,dx$ Converges $\displaystyle\sum_{n=1}^{\infty} f(n)$ Converges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad f(x) = 1/x$$

$$\int_1^{\infty} \frac{1}{x}\,dx \text{ diverges.}$$

The Series $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad f(x) = 1/x^2$$

$$\int_1^{\infty} \frac{1}{x^2}\,dx \quad \text{converges.}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>converges.</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad f(x) = \frac{1}{x^p}$$

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$ converges only when $p > 1$

So

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ converges if $p > 1$

diverges if $p \le 1$.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

Converges if $p>1$
diverges if $p \le 1$

$$f(x) = \frac{1}{x(\ln x)^p}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\, dx$$

$u = \ln x$
$du = \frac{dx}{x}$

$$= \int_{\ln 2}^{\infty} \frac{du}{(u)^p}$$

Converges when $p>1$
diverges if $p \le 1$.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$ converges

$$\sum_{n=2}^{\infty} \frac{1}{n\left[\ln n\right]^{1/2}}$$ diverges.