Lesson 22 11.3 and 11.4

Integral test and Comparison tests

Integral test:

$$\sum_{n=1}^{\infty} \underbrace{f(n)}_{a_n} \quad , \quad a_n = f(n)$$

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad ; \quad f(x) = 1/x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad ; \quad f(x) = \frac{1}{x^2}$$

(1) $f(x) > 0$ ✓

(2) $f(x)$ is decreasing ✓

(3) $\lim_{x \to \infty} f(x) = 0$ ✓

Then

$\sum_{n=1}^{\infty} f(n)$ converges if and only if

$\int_1^{\infty} f(x)\,dx$ converges.

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \, ; \quad p > 0$$

$$f(x) = \frac{1}{x^p}$$

The series converges only if

$$\int_1^{\infty} \frac{dx}{x^p} \quad \text{converges}$$

$$\underline{p > 1}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$ converges if and only if $p > 1$.

$$f(x) = \frac{1}{x(\ln x)^p}.$$ Set $u = \ln x$

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^{\infty} \frac{du}{u^p} = \int_{\ln 2}^{\infty} u^{-p}\,du$$

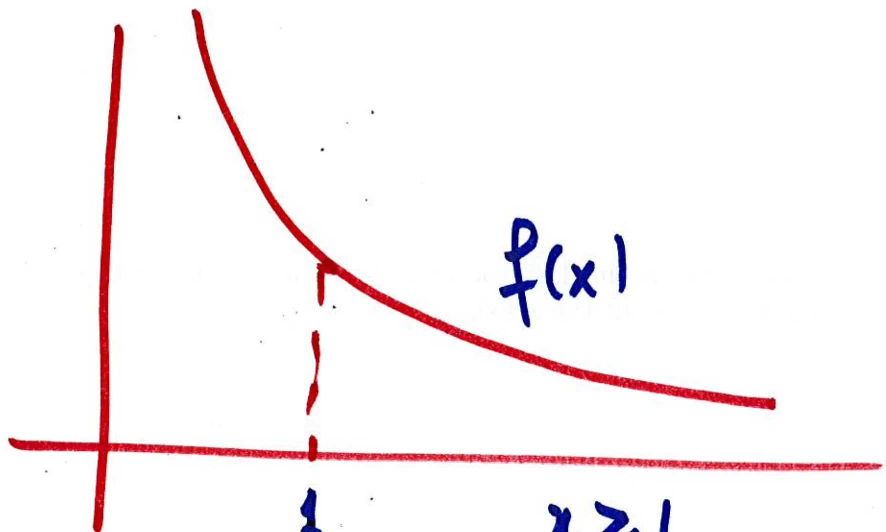
$$= +\frac{1}{1-p}\,u^{1-p}\Big|_{\ln 2}^{\infty} = \frac{1}{p-1}(\ln 2)^{1-p}$$

if $p > 1$

diverges if $p \leq 1$.

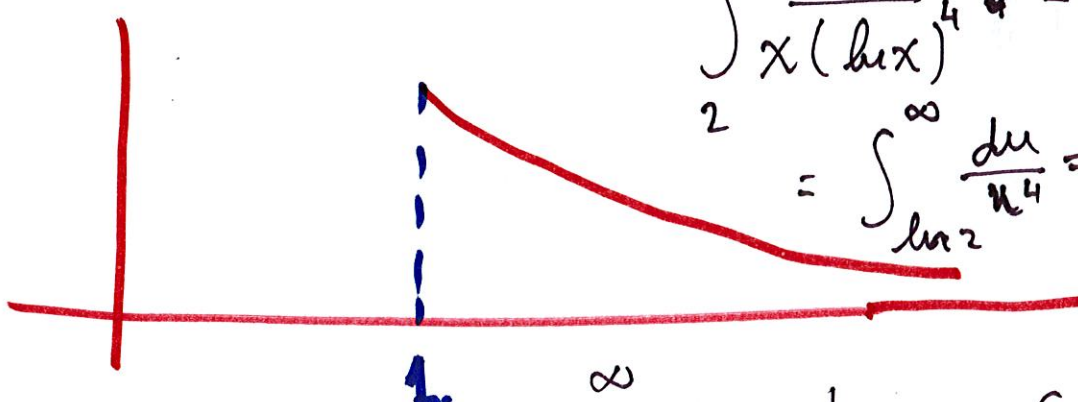
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$$

$$f(x) = \frac{1}{x(\ln x)^4}$$



Set $u = \ln x$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^4} = \int_{\ln 2}^{\infty} \frac{du}{u^4} = \frac{1}{3}(\ln 2)^{-3}$$



So $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ converges.

$$\sum_{n=1}^{\infty} n e^{-n}$$

$$\underline{f(x) = x e^{-x}} \quad , \quad x > 1$$

$$f'(x) = e^{-x} - x e^{-x}$$
$$= e^{-x}(1-x) < 0 \text{ if } x > 1$$

$f'(x) < 0$, $f(x)$ is decreasing.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x e^{-x}$$

$$= \lim_{x\to\infty} \frac{x}{e^{x}} = \lim_{x\to\infty} \frac{1}{e^{x}} = 0.$$

$$\sum_{n=1}^{\infty} n e^{-n}$$ converges

if and only if

$$\int_1^{\infty} x e^{-x} \, dx$$

converges.

$$\int \underbrace{x}_{u} \underbrace{e^{-x} dx}_{dv} = -xe^{-x} + \int e^{-x} dx$$

$$= -xe^{-x} - e^{-x}.$$

$du = dx$, $dv = e^{-x} dx$

$u = x$ $\quad$ $v = -e^{-x}$

$$\int_1^{\infty} x e^{-x}\,dx = -\left[x e^{-x} + e^{-x}\right]_1^{\infty}$$

$$= -\left[0 - e^{-1} - e^{-1}\right] = 2e^{-1}.$$

<u>Conclusion:</u> $\displaystyle\int_1^{\infty} x e^{-x}\,dx$

is finite, so

$\displaystyle\sum_{n=1}^{\infty} n e^{-n}$ converges.

$$\sum_{n=1}^{\infty} \frac{n^4}{n^5+10} \quad \text{diverges.}$$

$$f(x) = \frac{x^4}{x^5+10}$$

$$\int_1^{\infty} \frac{x^4}{x^5+10}\,dx = \int_1^{\infty} \frac{du}{5u}$$

$$\boxed{u = x^5+10,\ du = 5x^4\,dx}$$

$$= \frac{1}{5} \ln u \Big|_1^{\infty} \text{ : diverges.}$$

We have learned so far:

$$\sum_{n=1}^{\infty} r^n$$

Geometric Series

converges if $|r|<1$

diverges if $|r| \geq 1$

INTEGRAL TEST

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

converges if $p>1$

diverges if $p \leq 1$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges if $p>1$

diverges if $p \leq 1$.

The comparison tests

Are Σ's used to compare two series, one we already know with one we do not know.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+5}$ converge?

The Comparison test;

If $a_n \geq 0$ and $b_n \geq 0$

$n = 1, 2, 3, \ldots$ and

$$\boxed{a_n \geq b_n}$$

So

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n > 0$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{n+10}{n\sqrt{n}} \quad \text{diverges.}$$

$$n+10 > n$$

$$\frac{n+10}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad 1/2 = p$$

$$p = 1/2 < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+5}$$

$$n^2 + \underbrace{3n+5} \geq n^2$$

$$\frac{1}{n^2+3n+5} \leq \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+5}$ Converges.

The Limit Comparison Test:

$$a_n > 0 \,, \quad b_n > 0$$

$n = 1, 2, \ldots$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

If $\quad L \neq 0, \quad L \neq \infty$

Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges.

Or in other words, Either Both Series Converge or Both Series diverge.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+8n+9}{n^6+3n^4+4n^3}$

$$\boxed{a_n = \frac{n^5+8n+9}{n^6+3n^4+4n^3}} = \frac{n^5\left(1+8/n^4+9/n^5\right)}{n^6\left(1+3/n^2+4/n^3\right)}$$

$$= \frac{1}{n}\left(\frac{1+8/n^4+9/n^5}{1+3/n^2+4/n^3}\right).$$

$$\frac{a_n}{1/n} = \frac{1 + 8/n^4 + 9/n^5}{1 + 3/n^2 + 4/n^3}$$

$$\lim_{n \to \infty} \frac{a_n}{1/n} = 1$$

$\sum_{n=1}^{\infty} a_n$ converges if and only if

if

$\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so

$\sum_{n=1}^{\infty} a_n$ diverges