Lesson 23 Section 11.4

The Comparison Tests

Short Review

Geometric Series.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r|<1$$

Diverges if $|r| \geq 1$.

$$\sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n = \frac{1}{1-\tfrac{2}{3}} = 3.$$

The p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{converges if } p>1, \text{ diverges if } p\leq 1$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad \text{converges if } p>1, \text{ diverges if } p\leq 1$$

The integral test:

$$\sum_{n=1}^{\infty} f(n) \qquad a_n = f(n)$$

$$f(x) = \frac{1}{x^p}, \quad f(x) = \frac{1}{x(\ln x)^p}$$

If $f(x) \geq 0$, $f(x)$ is decreasing for $x \geq 1$, $\lim_{x \to \infty} f(x) = 0$

$\sum_{n=1}^{\infty} f(n)$ converges if and only

if

$$\int_1^{\infty} f(x)\, dx$$

Converges.

$$\int_1^{\infty} \frac{dx}{x^p}$$ converges if $p > 1$

diverges if $p \leq 1$.

Next we want compare two series.

The Comparison test

$a_n \geq b_n \geq 0 \quad n = 1, 2, 3, \ldots$

If $\sum_{n=1}^{\infty} a_n$ converges, then

$\sum_{n=1}^{\infty} b_n$ also converges.

If $\sum_{n=1}^{\infty} b_n$ diverges, then

$\sum_{n=1}^{\infty} a_n$ diverges.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n}{n^4}$

$$\underbrace{\frac{n^3+n}{n^4}}_{a_n} > \frac{n^3}{n^4} = \underbrace{\frac{1}{n}}_{b_n}$$

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n}{n^4}$ also diverges.

The Limit Comparison Test:

$a_n > 0, \quad b_n > 0$

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$

$L \neq 0, \quad L \neq \infty$

Then either $\displaystyle\sum_{n=1}^{\infty} a_n$ $\underline{\underline{\text{and}}}$

$\displaystyle\sum_{n=1}^{\infty} b_n$ converge

OR

$\displaystyle\sum_{n=1}^{\infty} a_n$ $\underline{\underline{\text{and}}}$ $\displaystyle\sum_{n=1}^{\infty} b_n$ diverge

Examples:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

Step 1: $a_n = \dfrac{1}{\sqrt{n}-1}$

Observe the behavior of a_n as $n \to \infty$.

as $n \to \infty$ $a_n \sim \dfrac{1}{\sqrt{n}}$

So $\dfrac{1}{\sqrt{n}}$ is the series you want to compare a_n to.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} = \frac{\sqrt{n}}{\sqrt{n}-1} = 1.$$

$$a_n = \frac{1}{\sqrt{n}-1}, \quad b_n = \frac{1}{\sqrt{n}}$$

<u>Conclusion</u>: Either

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

Converge or

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

diverge.

We know that

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$$

diverges.

So $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ diverges.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$.

When n is large $\dfrac{6^n}{5^n-1} \sim \dfrac{6^n}{5^n}$.

$$a_n = \frac{6^n}{5^n - 1}, \quad b_n = \frac{6^n}{5^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6^n}{5^n-1} \cdot \frac{5^n}{6^n}$$

$$= \lim_{n\to\infty} \frac{5^n}{5^n - 1} = 1.$$

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$$

diverges.

<u>Conclusion</u>. $\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$ diverges.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$$

<u>Question</u>: For what values of p does

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^p}\right)$$

converge?

$$\sum_{n=1}^{\infty} \sin\left(1/n\right) \; ; \; \sum_{n=1}^{\infty} \sin\left(1/n^2\right)$$

We know that $\sin x \sim x$

as $x \to 0$

$$\sin\left(\frac{1}{n}\right) \sim 1/n$$

$$\sin\left(1/n^p\right) \sim 1/n^p$$

Let $a_n = \sin\left(1/n^p\right)$, $b_n = \frac{1}{n^p}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n^p)}{1/n^p}$$

Set $x = 1/n^p$

$$= \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

$$\sum_{n=1}^{\infty} \sin\left(1/n^p\right) \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converge or

$$\sum_{n=1}^{\infty} \sin\left(1/n^p\right) \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverge.

When $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

So $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^p}\right)$ converges.

When $p \leq 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

So $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^p}\right)$ diverges.

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.

$$\sum_{n=1}^{\infty} \frac{n^{1/3}}{\sqrt{n^3+4n+3}}$$

$$\sqrt{n^3+4n+3} \sim n^{3/2}$$

$$\frac{n^{1/3}}{\sqrt{n^3+4n+3}} \sim \frac{n^{1/3}}{n^{3/2}} = n^{1/3-3/2}$$

$$= n^{\frac{2-9}{6}}$$

$$= n^{-7/6} = \frac{1}{n^{7/6}}$$

$$\frac{n^{1/3}}{\sqrt{n^3+4n+3}} \sim \frac{1}{n^{7/6}}.$$

$$\sum_{n=1}^{\infty} \left(\sqrt{n^3+2} - \sqrt{n^3+1} \right).$$

$$a_n = \left(\sqrt{n^3+2} - \sqrt{n^3+1} \right) \times \frac{\left(\sqrt{n^3+2} + \sqrt{n^3+1} \right)}{\sqrt{n^3+2} + \sqrt{n^3+1}}$$

Remember that $(a+b)(a-b) = a^2 - b^2$

$$a_n = \frac{(n^3+2) - (n^3+1)}{\sqrt{n^3+2} + \sqrt{n^3+1}}$$

$$a_n = \frac{1}{\sqrt{n^3+2} + \sqrt{n^3+1}}.$$

$$\sum_{n=1}^{\infty} \left(\sqrt{n^3+2} - \sqrt{n^3+1}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2} + \sqrt{n^3+1}}.$$

$$\sim \sum_{n=1}^{\infty} \frac{1}{2n^{3/2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p = \frac{3}{2} > 1$$

Converges.