

Series

$$\sum_{n=1}^{\infty} a_n.$$

Test 0 : If $\lim_{n \rightarrow \infty} a_n$

either does not exist or

$$\lim_{n \rightarrow \infty} a_n = L \neq 0$$

The series diverges.

However if $\lim_{n \rightarrow \infty} a_n = 0$

then we have to analyze the series a little closer.

Geometric Series

$$(1) \quad \sum_{n=1}^{\infty} r^n$$

Converges if $|r| < 1$

Diverges if $|r| \geq 1$.

$$(2) \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{Converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{array}$$

$$(3) \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad \begin{array}{l} \text{Converges if } p > 1 \\ \text{diverges if } p \leq 1. \end{array}$$

Comparison tests.

$$\sum_{n=1}^{\infty} a_n, \quad \& \sum_{n=1}^{\infty} b_n.$$

(1) If 0 $\leq a_n \leq b_n$

$\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges

(2) If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges

The limit comparison: If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0, \quad L \neq \infty$$

Then both series converge or
both diverge.

Alternating Series..

Example. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$S_N = \sum_{n=1}^N (-1)^{n-1} \underline{\underline{\frac{1}{n}}} \quad b_n = \frac{1}{n}$$

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

odd terms:

$$0 < S_1 = \underline{1}$$

$$0 < S_3 = 1 - \frac{1}{2} + \frac{1}{3} < S_1$$

$$0 < S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} < S_3$$

$$0 < S_7 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} < S_5$$

Conclusion S_N converges

when N is odd.

Same argument works for

N even.

So S_N converges

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \text{ Converges.}$$

Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n.$$

(1) $b_n > 0$

When does an alternating series converge?

Particular case

(2) If b_n is decreasing.

$$b_1 > b_2 > b_3 > b_4 > \dots$$

(3) $\lim_{n \rightarrow \infty} b_n = 0.$

If (1), (2) and (3)

happen

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

Converges.

Examples :

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}} \quad \text{Converges.}$$

$$b_n = \frac{1}{\sqrt{n}}.$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$$

$$b_n = \sqrt{n+1} - \sqrt{n}.$$

(1) Is $b_n > 0$

(2) Is b_n decreasing?

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$$

$$x+1 > x, \quad \sqrt{x+1} > \sqrt{x}$$
$$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$$

Conclusion: ~~18~~

$f(x)$ is decreasing.

$$f(n) = b_n = \sqrt{n+1} - \sqrt{n}$$

is decreasing.

$$(3) \quad \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$$

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$$

Conclusion: $\sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$

Converges.

$$|S_{\underline{N}} - S| \leq b_{\underline{N+1}}$$

Example:

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$S_{1000} = \sum_{n=1}^{1000} (-1)^{n-1} \frac{1}{n} \quad b_n = \frac{1}{n}$$

$$\boxed{|S - S_{1000}| \leq \frac{1}{1001} \quad N=1000}$$

Estimating the Sum of
an alternate series.

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

(1) $b_n > 0$, (2) b_n is decreasing

(3) $\lim_{n \rightarrow \infty} b_n = 0$

$$S_N = \sum_{k=1}^N (-1)^{k-1} b_k.$$

Can we estimate $|S - S_N|$

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!}$$

$$S_N = \sum_{n=1}^N (-1)^{n-1} \frac{1}{n!}$$

Question: Find the
Smallest N such that

$$|S - S_N| < 10^{-5}.$$

$$|S - S_N| \leq b_{N+1}$$

In this case $b_n = 1/n!$

$$|S - S_N| \leq \frac{1}{(N+1)!}$$

$$\frac{1}{(N+1)!} < 10^{-5}$$

$$(N+1)! > 10^5.$$

$$\underline{N=1} \quad (N+1)! = 2! = 2$$

$$\underline{N=5}: \quad (N+1)! = 6! = 6 \times 5 \times 3 \times 2$$

$$\underline{N=9}$$

$$(N+1)! = 10! = \underline{10 \times 9} \times 8 \times 7 \times 6 \times 5 \\ \times 3 \times 2 \times 1 =$$

$$N=10$$

$$(N+1)! = (11)!$$

$$= 11 \times 10 \times 9 \times \dots \times 1 =$$