Lesson 25 Section 11.6

Absolute Convergence and Ratio Test.

Example:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$$ Converges.
because it is an alternating series

However

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges.

If
$$\sum_{n=1}^{\infty} a_n \quad \text{converges, but}$$

$$\sum_{n=1}^{\infty} |a_n| \quad \text{diverges}$$

We say that $\sum_{n=1}^{\infty} a_n$ converges

CONDITIONALLY

Example. $a_n = (-1)^{n-1} \frac{1}{n}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \quad \text{converges conditionally}$$

because $|a_n| = \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{\ln n}$$

Converges because it is of the form

$$\sum_{n=2}^{\infty} (-1)^{n-1} b_n \quad , \quad b_n = \frac{1}{\ln n}$$

(1) $b_n > 0$

(2) b_n is decreasing

(3) $\lim_{n \to \infty} b_n = 0$.

However $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

If $\sum_{n=1}^{\infty} |a_n|$ converges

We say that

$$\sum_{n=1}^{\infty} a_n$$

converges <u>ABSOLUTELY</u>

<u>Ex:</u> $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$, converges

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$$ converges absolutely

Remark: If $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

Reason:

$$0 \leq |a_n| - a_n \leq \underbrace{2|a_n|}$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, by the comparison test $\sum_{n=1}^{\infty} (|a_n| - a_n)$ converges.

Then

$$\sum_{n=1}^{\infty} (a_n - |a_n|) \text{ converges.}$$

Then

$$\sum_{n=1}^{\infty} (a_n - |a_n|) + \sum_{n=1}^{\infty} |a_n|$$

$$= \sum_{n=1}^{\infty} (a_n - \cancel{|a_n|} + \cancel{|a_n|})$$

$$= \sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

Test For Absolute Convergence.

$$\sum_{n=1}^{\infty} a_n$$

The Ratio test

If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If $L < 1$, the series converges absolutely.

If $L > 1$,

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

If $L = 1$, The test is inconclusive.

Examples. The geometric series

$$\sum_{n=1}^{\infty} r^n$$

$a_n = r^n$, $a_{n+1} = r^{n+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{r^{n+1}}{r^n}\right| = |r|.$$

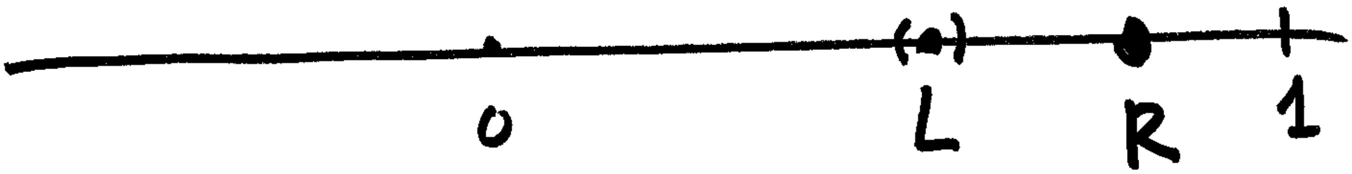
Converges if $|r| < 1$

Diverges if $|r| \geq 1$.

In fact the ratio test is a type of comparison test with the geometric series.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If $\underline{L<1}$.



$$\left| \frac{a_{n+1}}{a_n} \right| \sim L \text{ for } n \text{ large.}$$

$$\left| \frac{a_{n+1}}{a_n} \right| \leq R < 1$$

for $n \geq N$.

$$\left| \frac{a_{N+1}}{a_N} \right| \le R.$$

$$|a_{N+1}| \le R|a_N|$$

$$\left| \frac{a_{N+2}}{a_{N+1}} \right| \le R$$

$$|a_{N+2}| \le R\,|a_{N+1}| \le R^2 |a_N|$$

$$|a_{N+3}| \le R^3\,|a_N|$$

$$|a_{N+4}| \le R^4\,|a_N|$$

and so on.

$$\sum_{k=0}^{\infty} |a_{N+k}| \leq \sum_{k=0}^{\infty} R^k |a_N|$$

$$= |a_N| \sum_{k=0}^{\infty} \underline{\underline{R^K}}$$

Converges because $\underline{\underline{R < 1}}$.

Examples: Does

$$\sum_{n=1}^{\infty} n e^{-n} (-1)^{n-1}$$

converge absolutely?

$$a_n = n e^{-n} (-1)^{n-1}$$

$$|a_n| = n e^{-n} \quad |a_{n+1}| = (n+1) e^{-(n+1)}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1) e^{-(n+1)}}{n e^{-n}}$$

$$= \frac{n+1}{n} \cdot e^{-1}.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$$

$$= \lim_{n\to\infty} \frac{n+1}{n} e^{-1} = e^{-1} < 1$$

The series converges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^{10}}. \qquad a_n = \frac{n!}{n^{10}}$$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{10}}.$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{10}} \cdot \frac{n^{10}}{n!}$$

$$= \frac{(n+1)!}{n!} \frac{n^{10}}{(n+1)^{10}}$$

$$n! = n(n-1)(n-2)\dots 1.$$

$$(n+1)! = (n+1)\, n\, (n-1)(n-2)\dots 1$$

$$= (n+1)\, n!$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\, n!}{n!} \frac{n^{10}}{(n+1)^{10}}$$

$$= (n+1) \frac{n^{10}}{(n+1)^{10}}$$

$$\frac{a_{n+1}}{a_n} = \underbrace{n+1}\cdot \underbrace{\left(\frac{n}{n+1}\right)^{10}}_{\substack{\downarrow \\ 1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$$

Series diverges.

Take $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \; ; \; \boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad a_n = \frac{1}{n^2}$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 ; \; \boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1}$$