Lesson 26 Section 11.6

The Root Test

The Root Test

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

(1) If $L < 1$, $\sum_{n=1}^{\infty} |a_n|$ Converges

(2) If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

(3) If $L = 1$. The test is inconclusive.

Examples:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n .$$

$$a_n = \left(\frac{1}{2}+\frac{1}{n}\right)^n$$

$$a_n^{1/n} = \frac{1}{2}+\frac{1}{n}$$

$$\lim_{n\to\infty} a_n^{1/n} = \frac{1}{2} < 1.$$

The series converges.

$$\sum_{n=1}^{\infty} \left(\frac{3}{2} + \frac{1}{n}\right)^n$$ Diverges.

$$a_n = \left(\frac{3}{2} + \frac{1}{n}\right)^n$$

$$a_n^{1/n} = \frac{3}{2} + \frac{1}{n}$$

$$\lim_{n\to\infty} a_n^{1/n} = \frac{3}{2} > 1$$

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$ The series may converge or diverge.

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$a_n^{1/n} = 1 + \frac{1}{n} \qquad \lim_{n\to\infty} a_n^{1/n} = 1.$$

$$\sum_{n=1}^{\infty} \left(1+\tfrac{1}{n}\right)^n$$ The series diverges.

$$1+\tfrac{1}{n} > 1$$

$$\left(1+\tfrac{1}{n}\right)^n > 1.$$

$\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n$ is not equal to zero.

$$\lim_{n \to \infty} \left(1 + 1/n\right)^{n} = e.$$

In general

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^{n} = e^{a}.$$

$\cancel{\sum_{n=1}^{\infty} (1 +}$

33, 764

Section 11.7

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

Converges

$$a_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$a_n^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}$$

$$= \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

$$a_n^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}; \quad \lim_{n\to\infty} a_n^{1/n} = \frac{1}{e}.$$

$$< 1.$$

$$\sum_{n=1}^{\infty} a_n \;;$$

$$-n-1 < \ln|a_n| < -n$$

Does $\displaystyle\sum_{n=1}^{\infty} a_n$ converge absolutely?

This is the root test in disguise.

$$-n-1 < \ln|a_n| < -n$$

$$\left[e^{-n-1}\right]^{1/n} < |a_n|^{1/n} < \left[e^{-n}\right]^{1/n}$$

$$\left[e^{-n-1}\right]^{1/n} < |a_n|^{1/n} < e^{-1}.$$

$$e^{-\frac{n-1}{n}} < |a_n|^{1/n} < e^{-1}$$

$$e^{-1-\frac{1}{n}} < |a_n|^{1/n} < \underline{\underline{e^{-1}}}$$

$$\lim_{n \to \infty} e^{-1-\frac{1}{n}} = e^{-1}$$

$$\boxed{\lim_{n \to \infty} |a_n|^{1/n} = e^{-1}}$$

Conclusion $\sum_{n=1}^{\infty} a_n$ Converges

<u>ABSOLUTELY</u>

Strategy For Testing Series.

1st Thing to do

If $\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} |a_n| \neq 0$

The series diverges

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1} \text{. Diverges.}$$

$a_n = \frac{n^2-1}{n^2+1}$, $\lim_{n\to\infty} \left(\frac{n^2-1}{n^2+1}\right) = 1$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+8n+10}{\sqrt{2n^4+35n^2+8}}$$

$$\lim_{n\to\infty} \frac{n^2+8n+10}{\sqrt{2n^4+35n^2+8}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\cdot(1+8/n+10/n^2)}{\cancel{n^2}\sqrt{2+35/n^2+8/n^4}}$$

$$= \frac{1}{\sqrt{2}} \neq 0.$$

2nd thing to do

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} a_n = 0$$

See how a_n behaves with n and compare to series we already understand.

Bank of Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{converges if } p>1$$

$$\text{diverges if } p \leq 1$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad \text{converges if } p>1, \ \text{diverges if } p \leq 1.$$

$$\sum_{n=1}^{\infty} r^n \quad \text{converges if } |r|<1$$

$$\text{diverges if } |r| \geq 1.$$

$$\sum_{n=1}^{\infty} \frac{n^2+10n+8}{n^3+5n^2+10n} \text{. Diverges.}$$

When n is large

$$\frac{\boxed{n^2}+10n+8}{\boxed{n^3}+5n^2+10n} \sim \frac{n^2}{n^3} = \frac{1}{n}$$

$$a_n = \frac{n^2+10n+8}{n^3+5n^2+10n}$$

$$= \frac{\cancel{n^2}\left(1+\frac{10}{n}+8/n^2\right)}{n^{\cancel{3}}\left(1+5/n+\frac{10}{n^2}\right)}$$

$$a_n = \frac{1}{n} \quad \frac{1 + 10/n + 8/n^2}{1 + 5/n + \frac{10}{n^2}}$$

$$\frac{a_n}{1/n} = \frac{1 + 10/n + 8/n^2}{1 + 5/n + 10/n^2}$$

$$\boxed{\lim_{n \to \infty} \frac{a_n}{1/n} = 1}$$

The series diverges by limit Comparison with $\Sigma \frac{1}{n}$.

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

$$n^{1+\boxed{1/n}} \sim n \qquad (1/n \to 0)$$

$$\frac{1}{n^{1+1/n}} \sim \frac{1}{n}.$$

$$\frac{\frac{1}{n^{1+1/n}}}{1/n} = \frac{n}{n^{1+1/n}} = \frac{1}{n^{1/n}}$$

$$\boxed{\lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{1/n} = 1}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges by limit comparison

with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Ratio & Root Test

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1 \right)^n$$ Converges

$$a_n = \left(\sqrt[n]{2} - 1 \right)^n$$

$$(a_n)^{1/n} = \sqrt[n]{2} - 1 = 2^{\frac{1}{n}} - 1.$$

$$\lim_{n\to\infty} (a_n)^{1/n} = 2^0 - 1 = 0$$

$$\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!} \quad \text{Diverges.}$$

$$a_n = \frac{e^{n^2}}{n!}$$

$$a_{n+1} = \frac{e^{(n+1)^2}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{e^{\cancel{n^2}+2n+1}}{(n+1)\cdot n!} \quad \frac{n!}{e^{\cancel{n^2}}}$$

$$\frac{a_{n+1}}{a_n} = \frac{e^{2n+1}}{n+1} \to \infty \text{ as } n \to \infty$$