Lesson 27 Power Series

Basic Example

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

This is a sum of <u>powers of</u>

<u>x</u>

In general a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-b)^n$$

power series centered at b.

Starting point of many examples.

$$\boxed{\frac{1}{1-\underline{\underline{x}}} = \sum_{n=0}^{\infty} x^n , \quad |x|<1}$$

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

for $|x|<1$

$$\frac{1}{x} = \frac{1}{1-\underbrace{(1-x)}} = \sum_{n=0}^{\infty} (1-x)^n$$

provided $|x-1|<1$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n \qquad |1-x|<1$$

$$= \sum_{n=0}^{\infty} \left(-(x-1)\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n,$$

provided $|x-1|<1$

This is an example of a power series centered at 1.

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2+7}.$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{n^2+7} x^n.$$

<u>Question:</u> For what values of x does this converge?

We can use the ratio test for example.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2+7} x^n$$

Ratio Test: $\sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If $L < 1$, the series converges absolutely.

If $L > 1$, the series diverges

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2+7} x^n$$

$$a_n = \frac{2^n}{n^2+7} x^n$$

$$\lim_{n\to\infty} \frac{n^2+7}{(n+1)^2+7} =$$

$$\lim_{n\to\infty} \frac{\cancel{n^2}\left(1+7/n^2\right)}{\cancel{n^2}\left(\left(1+1/n\right)^2+7/n^2\right)} =$$

$$\lim_{n\to\infty} \frac{1+7/n^2}{\left(1+1/n\right)^2+7/n^2} = 1$$

$$a_{n+1} = \frac{2^{n+1}}{(n+1)^2+7} x^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)^2+7} \cdot x^{n+1} \cdot \frac{n^2+7}{2^n x^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = |2x| \cdot \frac{n^2+7}{(n+1)^2+7} \to 1 \text{ as } n \to \infty$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |2x|$$

<u>Conclusion:</u> If $2|x|<1$

$$\sum_{n=1}^{\infty} \frac{2^n \; x^n}{n^2+7}$$ converges.

ABSOLUTELY.

If $2|x|>1$, the series diverges.

What happens when $2|x|=1$?

We have two possibilities

$2x=1$, or $2x=-1$

$\boxed{x=1/2}$ or $\boxed{x=-1/2}$.

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2+7}$$

Substitute $x = 1/2$

$$= \sum_{n=1}^{\infty} \frac{2^n \cdot \left(\frac{1}{2}\right)^n}{n^2+7} = \sum_{n=1}^{\infty} \frac{1}{n^2+7}.$$

The series converges for $x = 1/2$.

Next Substitute $x = -1/2$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2+7} \cdot \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+7}.$$

The series converges for $x = -1/2$.

Conclusion: The

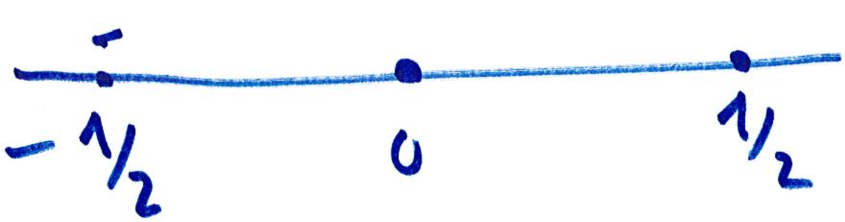
Series $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2+7}$

Converges if $|2x|<1$

but also if $2x=1$ or $2x=-1$

$$|2x|<1, \quad \boxed{2|x|<1}$$

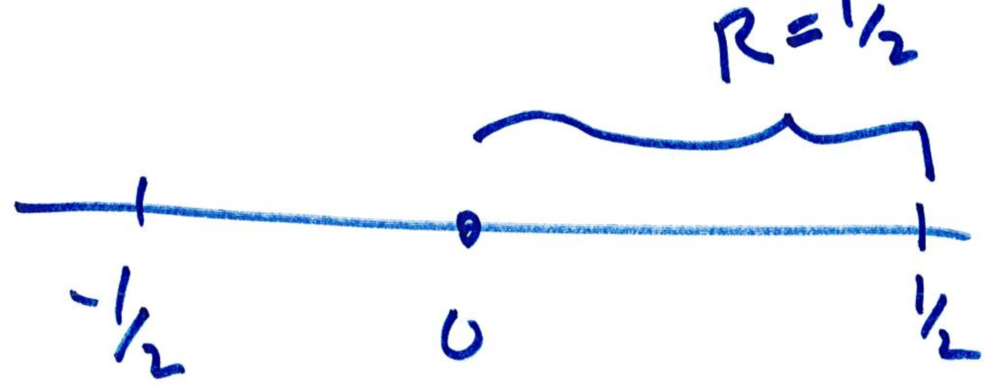
$$|x|<1/2$$



The series converges in

$[-\frac{1}{2}, \frac{1}{2}]$.

This is called the interval of convergence of the series.



$\frac{1}{2}$ = half of the length of the interval

= The Radius of Convergence.

$$\sum_{n=1}^{\infty} \frac{3^n (x-1)^n}{n^{1/3}} .$$

Determine the interval of convergence of this series

Step 1: Apply a convergence test

In this case I will use the root test.

Root test: $\sum_{n=1}^{\infty} a_n$,

$$\lim_{n \to \infty} |a_n|^{1/n} = L .$$

If $L < 1$, the series converges ABSOLUTELY

If $L > 1$, the series diverges.

If $L = 1$, the test is inconclusive.

In this particular case

$$\sum_{n=1}^{\infty} \frac{3^n (x-1)^n}{n^{1/3}}.$$

$$a_n = \frac{3^n (x-1)^n}{n^{1/3}}$$

$$|a_n| = 3^n \frac{|x-1|^n}{n^{1/3}}$$

$$|a_n|^{1/n} = \left[\frac{3^n \, |x-1|^n}{n^{1/3}} \right]^{1/n}$$

$$= \frac{3\,|x-1|}{n^{1/3n}}.$$

$$\boxed{\lim_{n\to\infty} |a_n|^{1/n} = 3\,|x-1|}$$

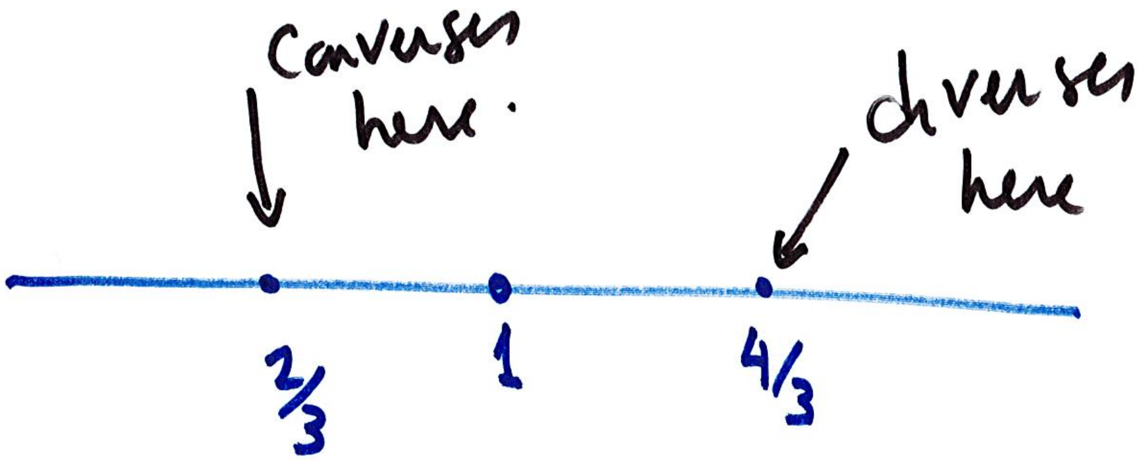
$$\boxed{\lim_{n\to\infty} n^{1/n} = 1.}$$

$$\lim_{n\to\infty} n^{1/3n} = 1$$

Converges Absolutely if

$$3|x-1| < 1$$

$$|x-1| < \frac{1}{3}$$



Analyze the cases $3(x-1) = 1$

and $3(x-1) = -1$

$$\sum_{n=1}^{\infty} \frac{3^n (x-1)^n}{n^{1/3}}$$

$$= \sum_{n=1}^{\infty} \frac{\left[3(x-1)\right]^n}{n^{1/3}}.$$

when $3(x-1) = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}, \quad \text{diverges}$$

when $3(x-1) = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}, \quad \text{Converges.}$$

The interval of convergence

is

$$\left[\frac{2}{3}, \frac{4}{3}\right)$$

Included

Excluded

Radius of convergence $= \frac{1}{3}$