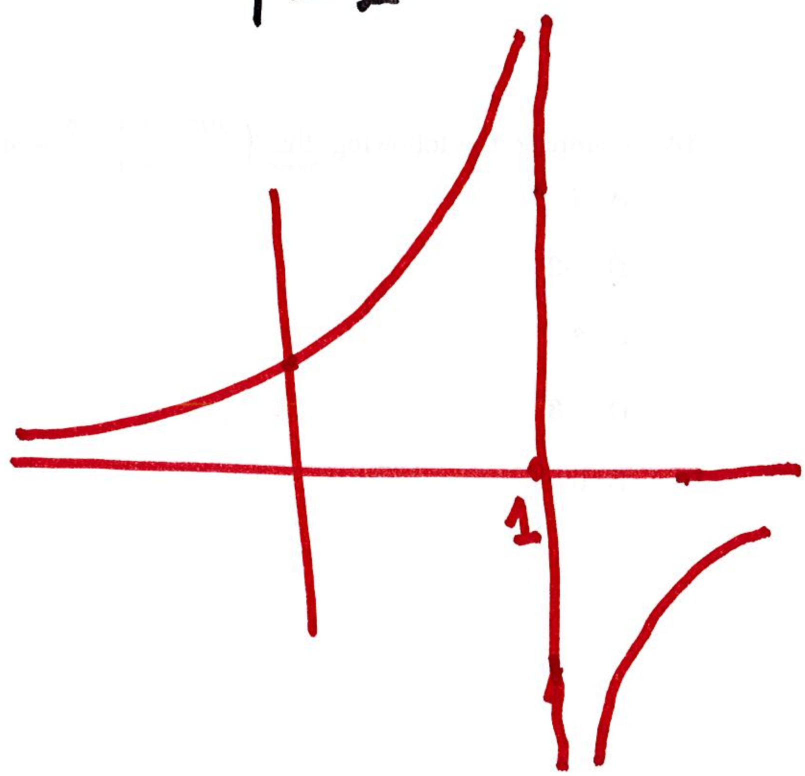
Lesson 28 Functions As Power Series Section 11.9

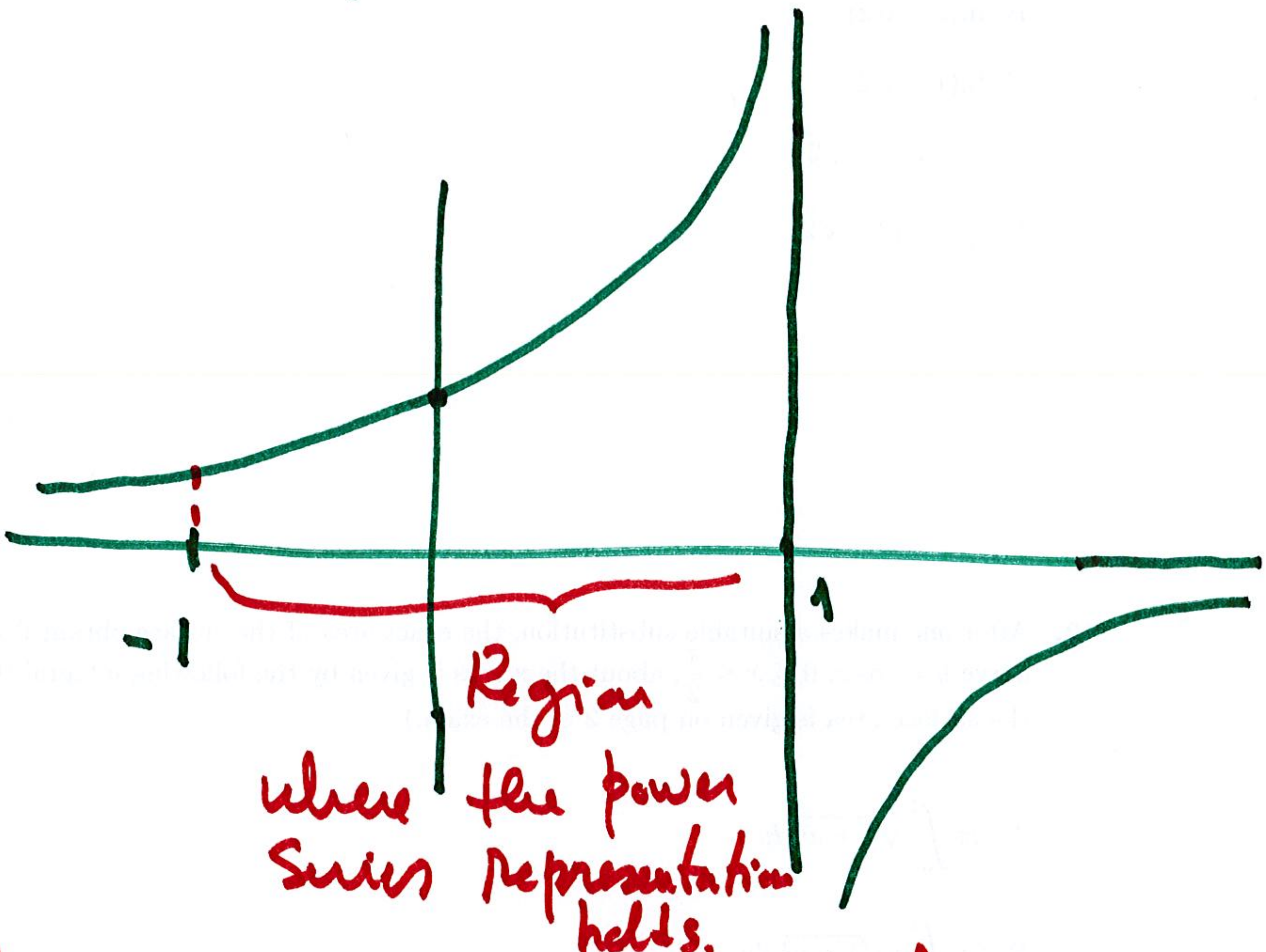
The geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

$f(x) = \frac{1}{1-x}$



$$f(x) = \frac{1}{1-x}$$



Thus function is well defined for all x, except $x=1$.

However when $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

When

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{in } |x-a| < R.$$

We say that $f(x)$ is represented by the power series in $|x-a| < R$.

Examples of functions Represented by Power Series.

Starting Point

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } \underline{|x|<1}$$

Variations of this example:

$$\frac{1}{5-x} = \frac{1}{5\left(1-\frac{x}{5}\right)} =$$

$$= \frac{1}{5} \cdot \frac{1}{1-x/5} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

$$\frac{1}{5-x} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$$

when $\left| \frac{x}{5} \right| < 1$

$$|x| < 5$$

$$(-5, 5)$$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$$

$|x^2| < 1$

$|x| < 1$

Theorem

If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a| < R$.

Then

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx + C$$

$$= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C.$$

$$f'(x) = \sum_{n=0}^{\infty} n\, c_n (x-a)^{n-1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}}$$

Provided $|x-a| < R$.

<u>Examples:</u>

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ , \quad |x| < 1$$

$$\int \frac{1}{1-x}\, dx = \sum_{n=0}^{\infty} \int x^n \, dx + C$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C.$$

$$\int \frac{1}{1-x} = -\ln|1-x|$$

Conclusion:

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C.$$

provided $|x| < 1.$

Substitute $x = 0$

$$-\ln 1 = \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} + C$$

$$\boxed{C = 0}$$

$$\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Provided $|x| < 1$.

In this interval $1-x > 0$

$$\boxed{\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \qquad |x| < 1}$$

Example:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

provided $|x| < 1$

$$\int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx + C.$$

$$\int \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C}$$

provided $|x| < 1$

To find c, set $x=0$ in the formula. We find $C=0$

So

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

$$\boxed{\text{Provided } |x|<1}$$

$$\tan \pi/6 = \frac{\sin \pi/6}{\cos \pi/6} = \frac{\cancel{1/2}}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\boxed{\pi/6 = \arctan 1/\sqrt{3}}$$

Substitute $x = 1/\sqrt{3}$ in the formula:

$$\arctan 1/\sqrt{3} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(1/\sqrt{3}\right)^{2n+1}}{2n+1}$$

$$\pi/6 = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n} \cdot \frac{1}{\sqrt{3}}}{2n+1}$$

$$= \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n (2n+1)}.$$

$$\boxed{\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n(2n+1)}}$$

<u>Question</u>: How many terms do you need to approximate π with an error $< 10^{-5}$.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Set $x = 1$ by Abel's theorem

$$\boxed{\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.}$$

Do similar things with derivatives.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \left((1-x)^{-1}\right)' = -(1-x)^{-2}\cdot(-1)$$

$$= (1-x)^{-2}.$$

$$\boxed{\begin{gathered}\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\,x^{n-1} \\ \text{Provided } |x|<1\end{gathered}}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

$$\frac{d}{dx} \frac{1}{1+x^2} = \frac{d}{dx} (1+x^2)^{-1}$$

$$= -(1+x^2)^{-2} \cdot 2x = -2x(1+x^2)^{-2}$$

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n 2n\, x^{2n-1}$$

$$\boxed{\frac{1}{(1+x^2)^2} = -\sum_{n=1}^{\infty} (-1)^n n\, x^{2n-2} \quad \text{Provided } |x| < 1}$$