Lesson 29   Section 11.10

Taylor and Maclaurin Series, part I

Exam 3   April 9, 6:30 P.M.

NOT IN ELLIOTT

ROOM ASSIGNMENTS
FORTHCOMING

A function $f(x)$ is represented by a power series if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + \sum_{n=1}^{\infty} c_n (x-a)^n$$

in $|x-a| < R$.

We have seen several examples, all based on the identity

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

<u>Ex:</u>

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

In this case $a=0$, $c_n = (-1)^n$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Again $\underline{a=0}$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \underbrace{(-1)^n}_{c_{2n}} x^{2n}$$

Even terms $c_{2n} = (-1)^n$.

<u>Example.</u>

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4$$

How do we find c_0, c_1, c_2, c_3 and c_4 from $p(x)$?

Substitute $x = a$

$$\underline{p(a) = C_0}$$

$$p'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3$$

Set $x = a$

$$p'(a) = C_1$$

$$p''(x) = 2C_2 + 6C_3(x-a) + 12C_4(x-a)^2$$

$$p''(a) = 2C_2 \; ; \quad C_2 = \frac{1}{2} p''(a).$$

Example: Let

$$p(x) = x^4 + 3x^3 + 6x + 1.$$

Express $p(x) = c_0 + c_1(x-3)$

$+ c_2(x-3)^2 + c_3(x-3)^3$

$+ c_4(x-3)^4.$

$\underline{a=3}$: $c_0 = p(3)$

$c_0 = 81 + 81 + 18 + 1 = 181$

$c_1 = p'(3)$ $\quad p'(x) = 4x^3 + 9x^2 + 6$

$p'(3) = 108 + 81 + 6 = 195.$

$$c_2 = \frac{p''(3)}{2!}$$

$$p''(x) = 12x^2 + 18x$$

$$p''(3) = 108 + 54$$
$$= 162.$$

$$c_2 = \frac{162}{2} = 81$$

$$c_3 = \frac{p''(3)}{3!}$$

$$p^{(3)}(x) = 24x + 18$$

$$p^{(3)}(3) = 90$$

$$c_3 = \frac{90}{6} = 15. \quad , \quad c_4 = 1.$$

Conclusion:

$$p(x) = x^4 + 3x^3 + 6x + 1$$

$$= \cancel{(x-3)^4}$$

$$= 181 + 195(x-3) + 81(x-3)^2$$

$$+ 15(x-3)^2 + (x-3)^4$$

$$6\, C_3 = p'''(a)$$

$$C_3 = \frac{p^{(3)}(a)}{6} = \frac{p^{(3)}(a)}{3!}$$

$$C_4 = \frac{p^{(4)}(a)}{24} = \frac{p^{(4)}(a)}{4!}$$

In general, if we have a polynomial of degree n

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots + C_n(x-a)^n.$$

$$C_0 = f(a)\,; \quad C_n = \frac{f^{(n)}(a)}{n!}$$

If a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a| < R$

The coefficients are given by

$$c_0 = f(a)$$

$$c_1 = f'(a)$$

$$\vdots$$

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

$$|x| < 1$$

$$\boxed{\underline{C_n = 1}}$$

$$f(x) = 1 + x + x^2 + x^3 + \cdots$$

$$f(0) = 1$$

$$\frac{f^{(n)}(0)}{n!} = 1 = C_n$$

$$f^{(n)}(0) = n!$$

The Taylor Series of a function.

$\boxed{\underline{\text{Example:}} \quad f(x) = e^x.}$

Define the Taylor Series of $f(x)$ centered at a

as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$f(x) = e^x$

Find the Taylor Series of

e^x centered at 1.

$f'(x) = e^x$; $f'(1) = e$

$f''(x) = e^x$, $f''(1) = e$

So the Taylor Series of

e^x centered at 1

is

$$\sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$$

Two questions:

(1) What is the interval of convergence of this series

(2) Does this series ~~[illegible]~~ represent e^x

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

in the interval of convergence?

Yes

(1) Use the ratio test:

$$a_n = \frac{e}{n!}(x-1)^n$$

$$a_{n+1} = \frac{e}{(n+1)!}(x-1)^{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!}\,|(x-1)| = \frac{1}{n+1}|x-1|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$$

The series converges for every x.
Interval of convergence $(-\infty, \infty)$.

$f(x) = \sin x$

Find the Taylor Series of $f(x)$ centered at 0.

Taylor Series centered at 0 are called Maclaurin Series.

$f(x) = \sin x$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \sin x$$

$$f(0) = \sin 0 = 0$$

$$f'(x) = \cos x, \quad f'(0) = 1.$$

$$f''(x) = -\sin x, \quad f''(0) = 0$$

$$f^{(3)}(x) = -\cos x; \quad f^{(3)}(0) = -1$$

$$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0.$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0$$

$$f^{(3)}(0) = -1, \quad f^{(4)}(0) = 0$$

$$f^{(5)}(0) = 1, \quad f^{(6)}(0) = 0$$

$$f^{(7)}(0) = -1$$