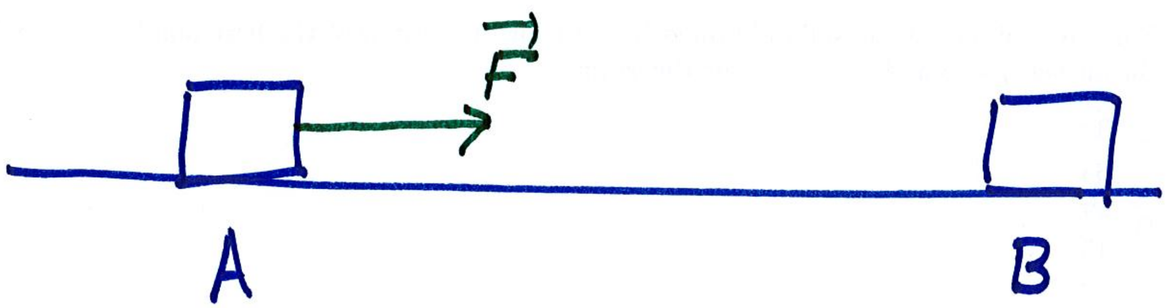
Lesson 3 Section 12.3

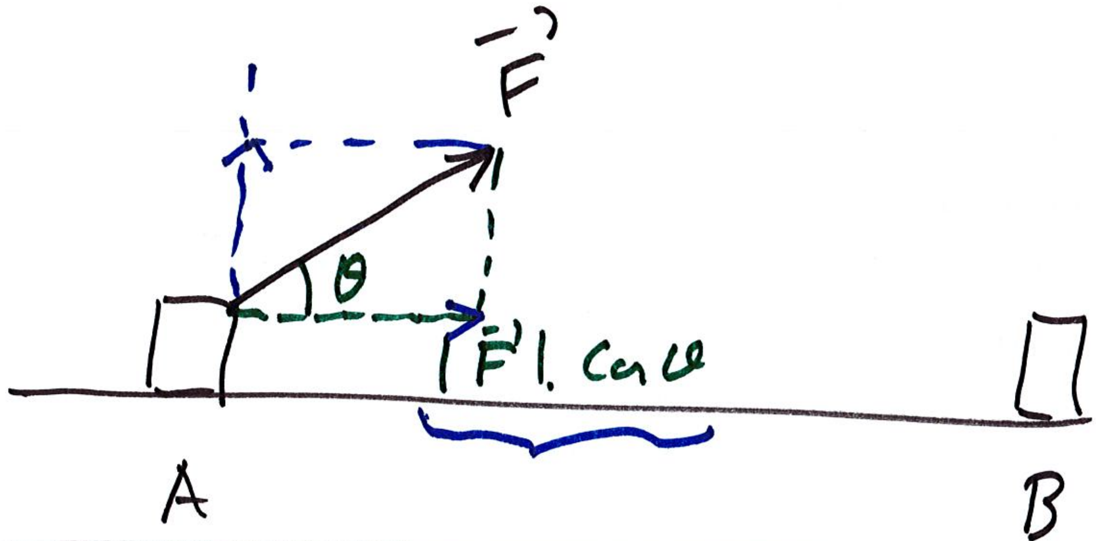
The Dot Product

Motivation



$$\text{Work} = |\vec{F}| \cdot |\overrightarrow{AB}|$$

$\vec{F}$ is parallel to $\overrightarrow{AB}$



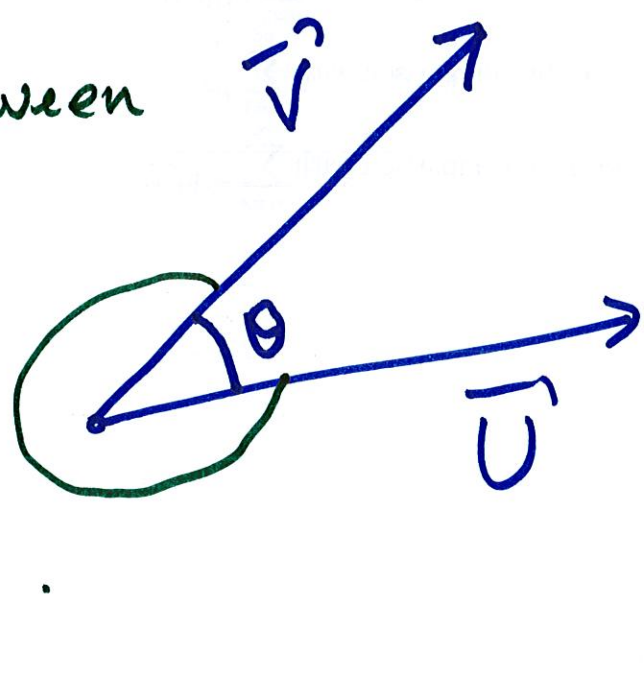
$$\boxed{\text{Work} = |\vec{AB}|.|\vec{F}|.\cos\theta}$$

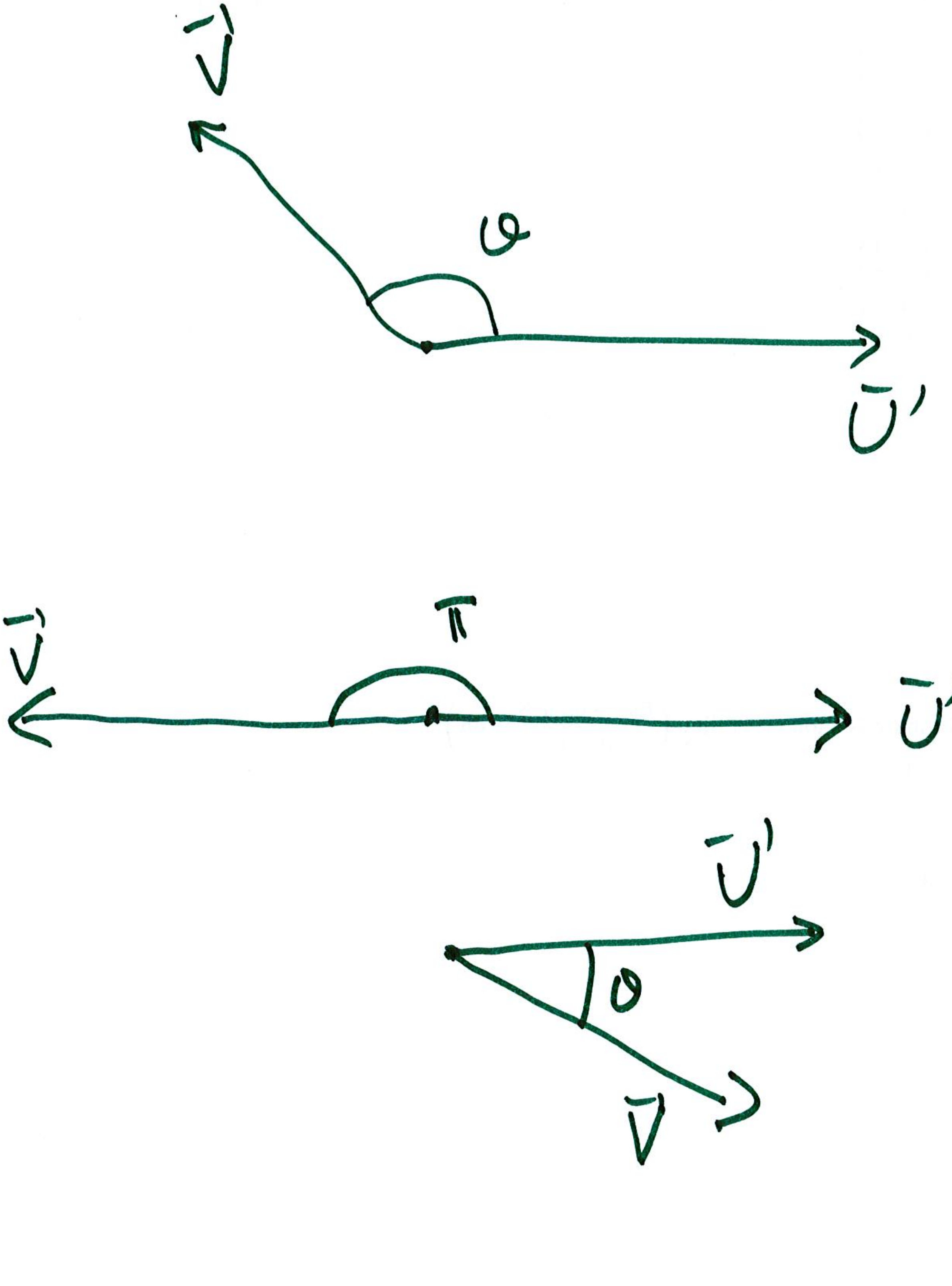
Mathematical Definition

θ = angle between $\vec{U}$ and $\vec{V}$

θ satisfies

$0 \leq \theta \leq \pi$.





θ
π
θ

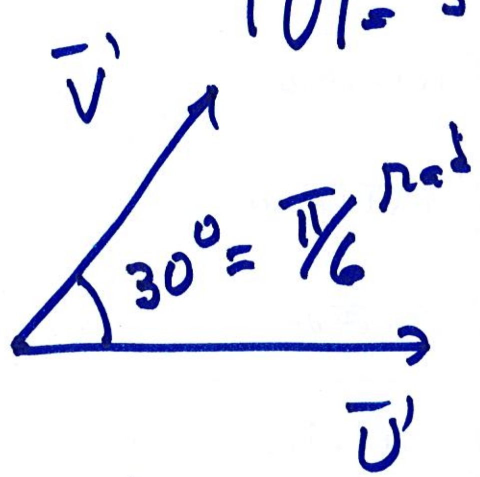
Given two vectors $\vec{U}$ and $\vec{V}$, the <u>Dot Product</u> between $\vec{U}$ and $\vec{V}$ is

$$\vec{U} \cdot \vec{V} = |\vec{U}| . |\vec{V}| . \cos\theta .$$

θ = angle between $\vec{U}$ and $\vec{V}$.

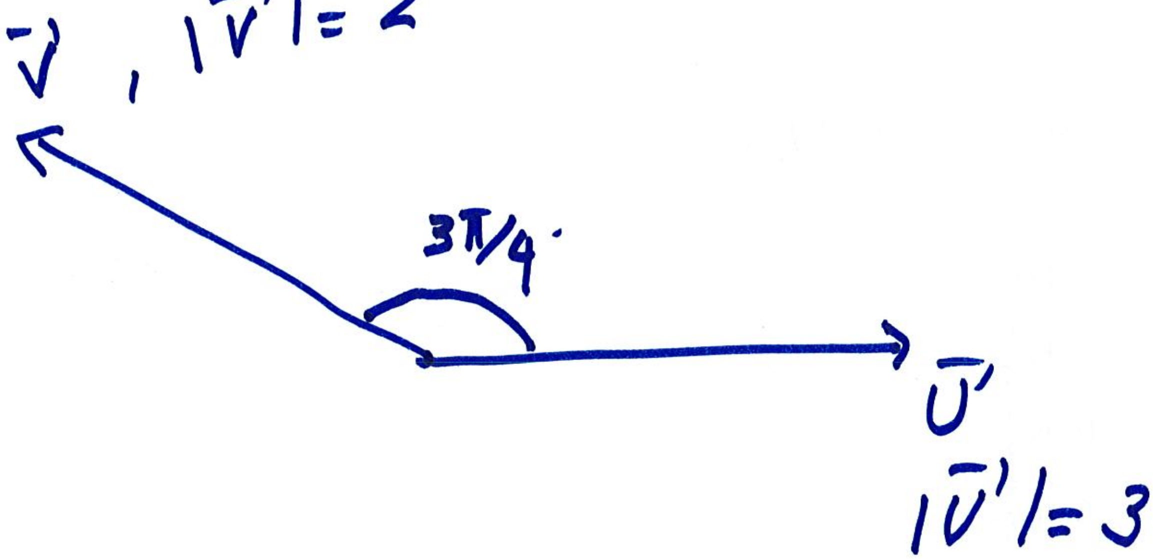
<u>Examples:</u>

$|\vec{U}| = 5$, $|\vec{V}| = 8$



$$\vec{U} \cdot \vec{V} = |\vec{U}| . |\vec{V}| . \cos \pi/6$$

$$= 5 . 8 . \frac{\sqrt{3}}{2} = 20\sqrt{3} .$$



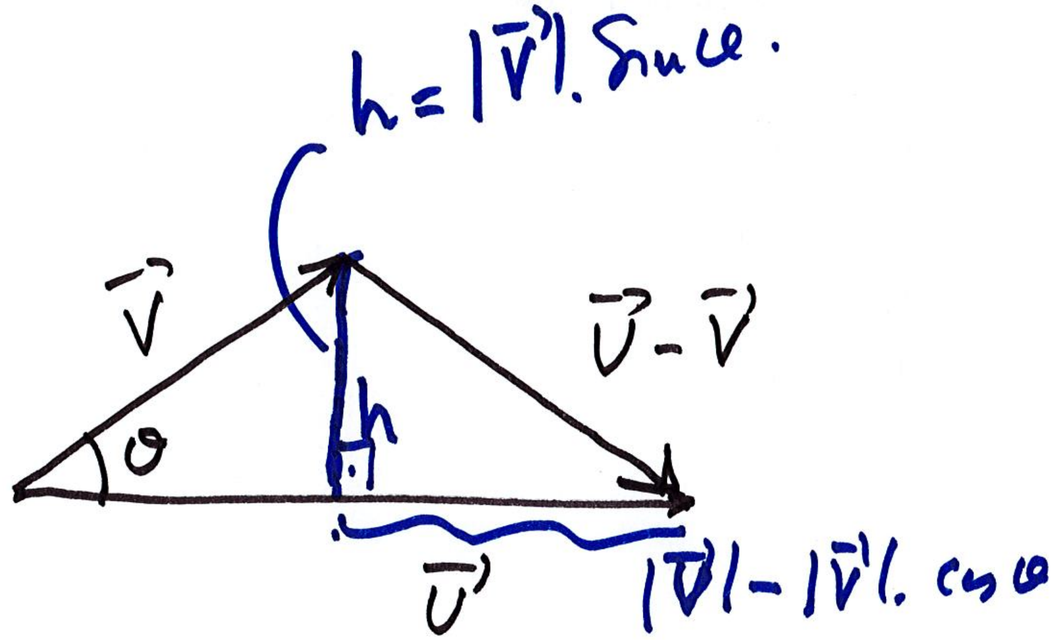
$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos 3\pi/4$$

$$= 6 \cdot \left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}.$$

We want to do this computation in coordinates.

$$\vec{u} = \langle a, b, c \rangle$$

$$\vec{v} = \langle x, y, z \rangle.$$



Compute $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$.

$\vec{u} = \langle a, b \rangle$, $\vec{v} = \langle x, y \rangle$.

$$|\vec{u} - \vec{v}|^2 = h^2 + \left(|\vec{u}| - |\vec{v}|\cos\theta\right)^2$$

$$|\vec{u} - \vec{v}|^2 = |v|^2 \sin^2\theta + \left(|\vec{u}| - |\vec{v}|\cos\theta\right)^2.$$

$$|\vec{U} - \vec{V}|^2 = |V|^2 \sin^2\theta +$$

$$|\vec{U}|^2 + |\vec{V}|^2 \cos^2\theta - 2|\vec{U}|.|\vec{V}|. \cos\theta.$$

$$= |\vec{U}|^2 + |\vec{V}|^2(\cos^2\theta + \sin^2\theta)$$

$$- 2|\vec{U}|.|\vec{V}|. \cos\theta$$

$$\boxed{|\vec{U} - \vec{V}|^2 = |\vec{U}|^2 + |\vec{V}|^2 - 2|\vec{V}|.|\vec{U}|. \cos\theta}$$

$$\vec{U} = \langle a, b \rangle \quad , \quad \vec{V} = \langle x, y \rangle$$

$$\vec{U} - \vec{V} = \langle a-x, b-y \rangle.$$

$$|\vec{U} - \vec{V}|^2 = (a-x)^2 + (b-y)^2.$$

$$|\vec{U} - \vec{V}|^2 = a^2 + x^2 - 2ax + b^2 + y^2 - 2yb$$

$$= \underbrace{(a^2 + b^2)}_{|\vec{U}|^2} + \underbrace{(x^2 + y^2)}_{|\vec{V}|^2} - 2(ax + yb)$$

$$\boxed{|\vec{U} - \vec{V}|^2 = |\vec{U}|^2 + |\vec{V}|^2 - 2(ax + yb).}$$

$$\cancel{|\vec{U}|^2} + \cancel{|\vec{V}|^2} - 2|\vec{U}|.|\vec{V}|\cos\theta$$

$$= \cancel{|\vec{U}|^2} + \cancel{|\vec{V}|^2} - 2(ax + yb).$$

$$\boxed{|\vec{U}|.|\vec{V}|.\cos\theta = ax + yb}$$

Conclusion:

$\vec{u} = \langle a, b \rangle, \quad \vec{v} = \langle x, y \rangle$

$$\boxed{\vec{u} \cdot \vec{v} = ax + by}$$

In 3d. $\vec{u} = \langle \underline{\underline{a}}, \underline{\underline{b}}, \underline{\underline{c}} \rangle$

$\vec{v} = \langle \underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{z}} \rangle$

$$\boxed{\vec{u} \cdot \vec{v} = ax + by + cz}$$

Example: $\vec{U} = \langle 1; 2; 1 \rangle$,

$\vec{V} = \langle -1, -2, 5 \rangle$

$$\vec{U} \cdot \vec{V} = -1 - 4 + 5 = 0$$

Find x such that

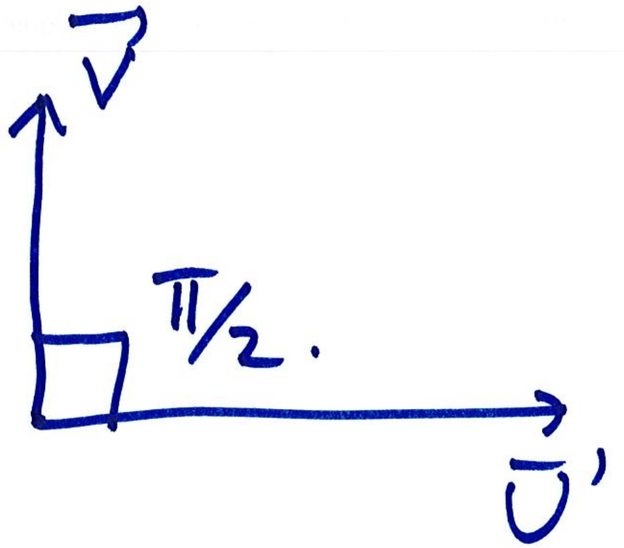
$\langle 1, 2, 3 \rangle$ and $\langle 0, 1, x \rangle$

are perpendicular.

$$\langle 1, 2, 3 \rangle \cdot \langle 0, 1, x \rangle = 2 + 3x = 0$$

$$x = -2/3.$$

Perpendicular (orthogonal) Vectors



$$\vec{U} \cdot \vec{V} = |\vec{U}| \cdot |\vec{V}| \cdot \cos\theta$$

$$= |\vec{U}| \cdot |\vec{V}| \cdot \underbrace{\cos\frac{\pi}{2}}_{=0}$$

$$\boxed{\vec{U} \cdot \vec{V} = 0}$$

Examples: Find $\vec{U} \cdot \vec{V}$

$\vec{U} = \langle 1,0,1 \rangle$, $\vec{V} = \langle 2,1,3 \rangle$.

$$\vec{U} \cdot \vec{V} = 1 \cdot 2 + 0 \cdot 1 + 1 \cdot 3$$

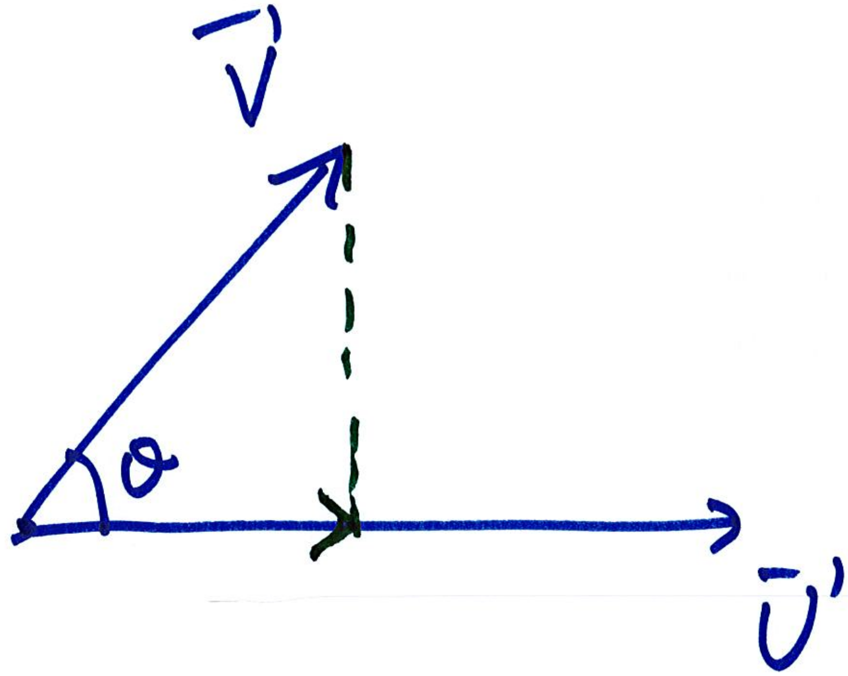
$$= 2 + 3 = 5.$$

Question: Find the angle between $\vec{U}$ and $\vec{V}$.

$$|\vec{U}| \cdot |\vec{V}| \cdot \cos\theta = \vec{U} \cdot \vec{V} = 5.$$

$$\cos\theta = \frac{5}{|\vec{U}| \cdot |\vec{V}|}.$$

Projections



1) <u>Component</u>. $\text{Comp}_{\vec{u}} \vec{v} = \underbrace{|\vec{v}| \cos\theta}$

2) <u>Projection</u>:

$$\text{Proj}_{\vec{u}} \vec{v} = \underbrace{|\vec{v}| \cdot \cos\theta} \cdot \frac{\vec{u}}{|\vec{u}|}$$

We want to do this computation in coordinates.

$$\vec{U}\cdot\vec{V} = |\vec{U}|.\underbrace{|\vec{V}|.\cos\theta}$$

$$\boxed{|\vec{V}|\cos\theta = \frac{\vec{U}\cdot\vec{V}}{|\vec{U}|}} = \mathrm{Comp}_{\vec{U}}\vec{V}$$

$$\mathrm{Proj}_{\vec{U}}\vec{V} = \frac{\vec{U}\cdot\vec{V}}{|\vec{U}|}\cdot\frac{\vec{U}}{|\vec{U}|}$$

$$= \frac{\vec{U}\cdot\vec{V}}{|\vec{U}|^2}\cdot\vec{U}.$$

$$|\bar{U}'| = \sqrt{1+0+1} = \sqrt{2}$$

$$|\bar{V}'| = \sqrt{4+1+9} = \sqrt{14}.$$

$$\cos\theta = \frac{5}{\sqrt{2}.\sqrt{14}} = \frac{5}{\sqrt{28}}.$$

$$\boxed{\cos\theta = \frac{5}{\sqrt{28}}}$$

Example: Let

$$\vec{U} = \langle 1, 2, 3 \rangle$$

$$\vec{V} = \langle 1, 1, 4 \rangle$$

Find $\text{Comp}_{\vec{U}} \vec{V}$, $\text{Proj}_{\vec{U}} \vec{V}$

$\longrightarrow$ $\text{Comp}_{\vec{V}} \vec{U}$, $\text{Proj}_{\vec{V}} \vec{U}$.

Solution: $$\text{Comp}_{\vec{U}} \vec{V} = \frac{\langle 1, 2, 3 \rangle \cdot \langle 1, 1, 4 \rangle}{|\langle 1, 2, 3 \rangle|}$$

$$= \frac{15}{\sqrt{14}}$$

$$\boxed{\text{Proj}_{\vec{U}} \vec{V} = \frac{15}{14} \langle 1, 2, 3 \rangle}$$

$$\operatorname{Comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|}$$

$$\operatorname{Comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

$$= \frac{15}{\sqrt{18}}$$

$$\operatorname{Proj}_{\vec{v}} \vec{u} = \frac{15}{18} \langle 1, 1, 4 \rangle$$