Lesson 30 Section 11.10

Continuation.

Taylor and Maclaurin Series

The Taylor series of a function $f(x)$ at a point a is defined

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We require that all derivatives of $f(x)$ are defined on an interval containing a.

<u>Example 1</u>: $f(x) = e^x$

$$f'(x) = e^x$$

$$f''(x) = e^x \; ; \dots \; f^{(n)}(x) = e^x$$

If I set $x = a$.

$$f^{(n)}(a) = e^a.$$

The Taylor series of e^x at a is

$$= \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n.$$

Use the ratio test to find the interval of convergence.

$$a_n = \frac{e^a}{n!}(x-a)^n$$

$$a_{n+1} = \frac{e^a}{(n+1)!}(x-a)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{e^a}}{(n+1)!}(x-a)^{n+1} \times \frac{n!}{\cancel{e^a}(x-a)^n}$$

$$= \frac{x-a}{n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1}|x-a|.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \quad \text{for every } x.$$

<u>Conclusion:</u> $\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$

Converges on $(-\infty, \infty)$.

$$\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$ converges to what?

Thus in fact converges to e^x

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n \quad \text{for all } x \text{ in } (-\infty, \infty)}$$

When $a=0$ this is called the Maclaurin Series of e^x

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.}$$

Variations of this series

For example, find the Maclaurin series of e^{3x}.

Solution: Just replace x with $3x$ in the Maclaurin series of e^x.

$$e^{3x} = \sum_{n=0}^{\infty} \frac{1}{n!} (3x)^n$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

This is an <u>alternating</u> series.

Compute

$$\int_0^1 x^5 e^{-x^2} dx$$

with an error $< 10^{-5}$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

$$x^5 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+5}.$$

$$\int_0^1 x^5 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n+5}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{1}{2n+6} = b_n$$

The estimation theorem.

$$b_{n+1} < 10^{-5}.$$

$$b_n = \frac{1}{n!\,(2n+6)}$$

$$b_{n+1} = \frac{1}{(n+1)!\,(2n+8)} < 10^{-5}$$

$\boxed{n = 6,}$ $$\frac{1}{7!\; 20} = \frac{1}{100{,}800} < 10^{-5}$$

$$\boxed{n = 6}$$

$$\int_0^1 x^5 e^{-x^2}\,dx = \frac{1}{6} - \frac{1}{8} + \frac{1}{20} - + \text{ First } 6 \text{ terms.}$$

Other applications:

Compute

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \underbrace{1 + x} + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \underbrace{\frac{x}{3!} + \frac{x^2}{4!} + \frac{x^4}{6!}} + \cdots$$

Conclusion:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}.$$

Taylor Series of $\sin x$ and $\cos x$.

The Maclaurin series of

$$\sin x = f(x)$$

We have to compute $f^{(n)}(0)$

for $n = 1, 2, \ldots$

$f(x) = \sin x \,, \quad f(0) = 0$

$f'(x) = \cos x \;; \quad f'(0) = 1$

$f''(x) = -\sin x \,, \quad f''(0) = 0$

$f^{(3)}(x) = -\cos x \,, \quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \,, \quad f^{(4)}(0) = 0$

Conclusion:

$f^{(2n)}(0) = 0$. all even order derivatives are equal to zero.

$f'(0) = 1, \quad f^{(3)}(0) = -1$

$f^{(5)}(0) = 1, \quad f^{(7)}(0) = -1$

$f^{(9)}(0) = 1, \quad f^{(11)}(0) = -1$

Odd numbers can be written

as $2n+1$.

$\underline{n=0}$: $\quad f^{(2n+1)}(0) = 1.$

$n=1 \quad f^{(2n+1)}(0) = -1$

$n=2 \quad f^{(2n+1)}(0) = f^{(5)}(0) = 1.$

$n=3 \quad f^{(2n+1)}(0) = f^{(7)}(0) = -1.$

$$f^{(2n+1)}(0) = (-1)^n.$$

$$f(x) = \sin x, \quad \underline{f^{(2n)}(0) = 0}$$

$$f^{(2n+1)}(0) = (-1)^n.$$

The Maclaurin Series of

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

for all x in $(-\infty, \infty)$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

Variations of these series

$$\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^3)^{2n+1}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n}$$

$$(\cos x)^2 = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right]^2$$

$$(\cos x)^2 = \frac{1+\cos 2x}{2}.$$

$$\frac{1+\cos 2x}{2} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(2x)^{2n}}{2}.$$

$$= (\cos x)^2.$$