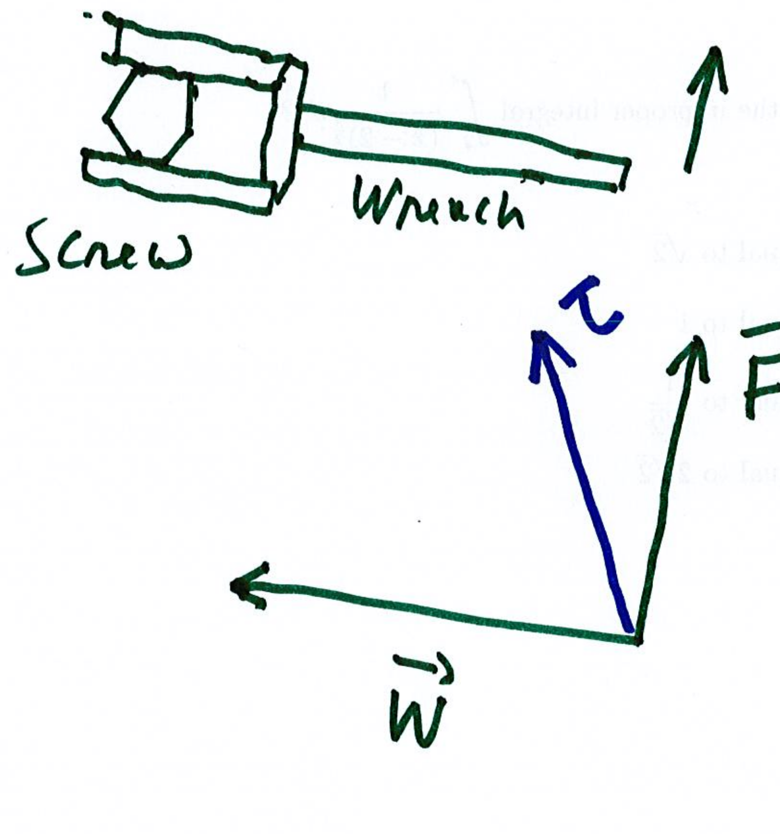
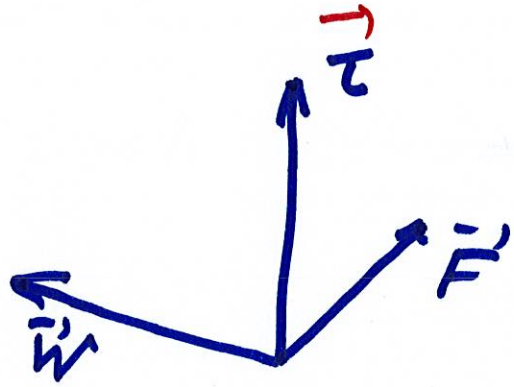
Lesson 4 Section 12.4

The Cross Product

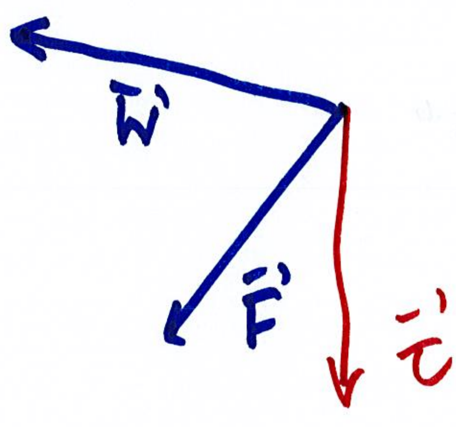
Physical Motivation: Torque τ



$$\vec{\tau} = \vec{F} \times \vec{W}$$



$\vec{\tau}$ is perpendicular to $\vec{F}$ and $\vec{W}$



The direction of $\vec{\tau}$ is given by the Right Hand Rule

The Cross Product

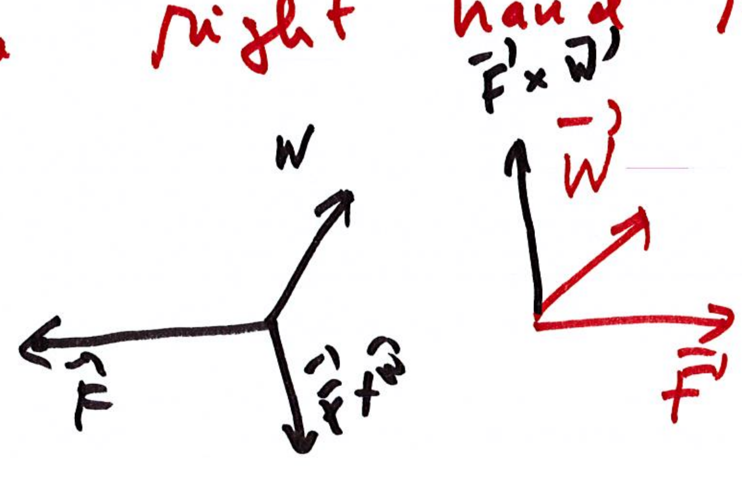
$\vec{F}$ and $\vec{W}$ are two vectors.

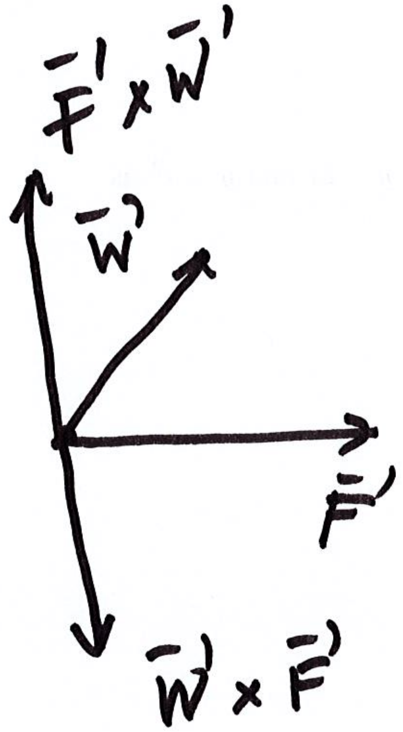
$\vec{F} \times \vec{W}$ = the cross product of $\vec{F}$ and $\vec{W}$.

$\vec{F} \times \vec{W}$ satisfies.

1) Is perpendicular to both $\vec{F}$ and $\vec{W}$.

2) $\vec{F} \times \vec{W}$ has direction given by the right hand rule.

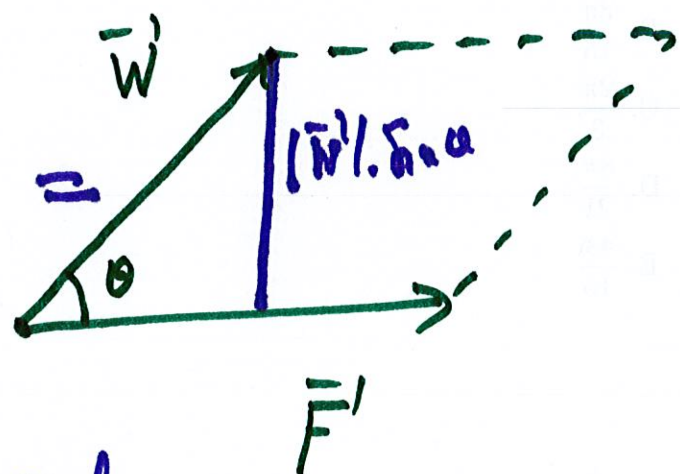




$$\boxed{\vec{F}' \times \vec{W}' = -\,\vec{W}' \times \vec{F}'}$$

3) $|\vec{W}' \times \vec{F}'| = |\vec{F}' \times \vec{W}'| = |\vec{F}'|.|\vec{W}'|.\sin\theta$

Area of the parallelogram defined by $\vec{F}'$ and $\vec{W}'$. =



We use determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Ex: $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

We would like to do this

computation in coordinates.

$$\vec{F} = \langle a_1, a_2, a_3 \rangle = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{W} = \langle b_1, b_2, b_3 \rangle = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}.$$

$$\vec{F} \times \vec{W} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,$$

$$a_1 b_2 - a_2 b_1 \rangle.$$

Better way to remember this

formula.

Example:

$$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$

$$+ 2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1.(1-0) - 3(0-2) + 2(-1)$$

$$= 1 + 6 - 2 = 5.$$

$$\vec{F} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{W} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}.$$

$$\vec{F} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: $\vec{F} = \vec{i} + \vec{j} + 3\vec{k}$

$$\vec{W} = \vec{i} - \vec{j}.$$

$$\vec{F} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \bar{i}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}$$

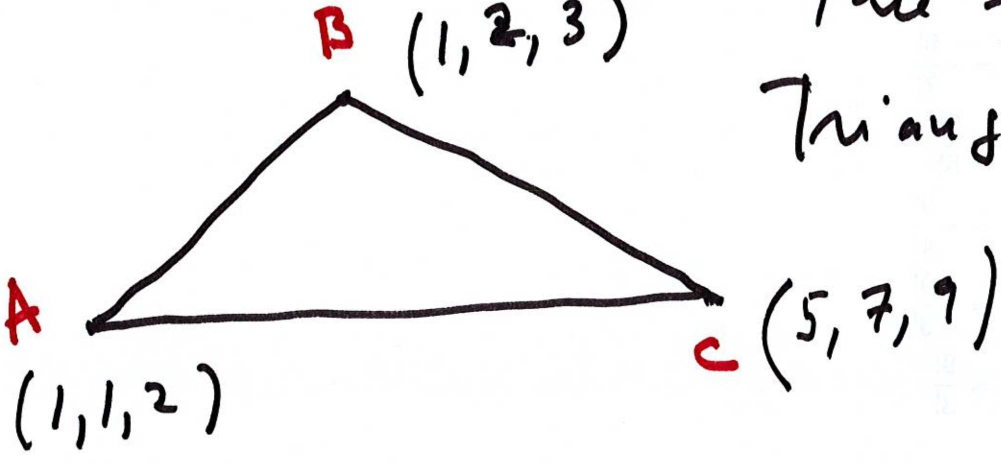
$$+ \bar{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

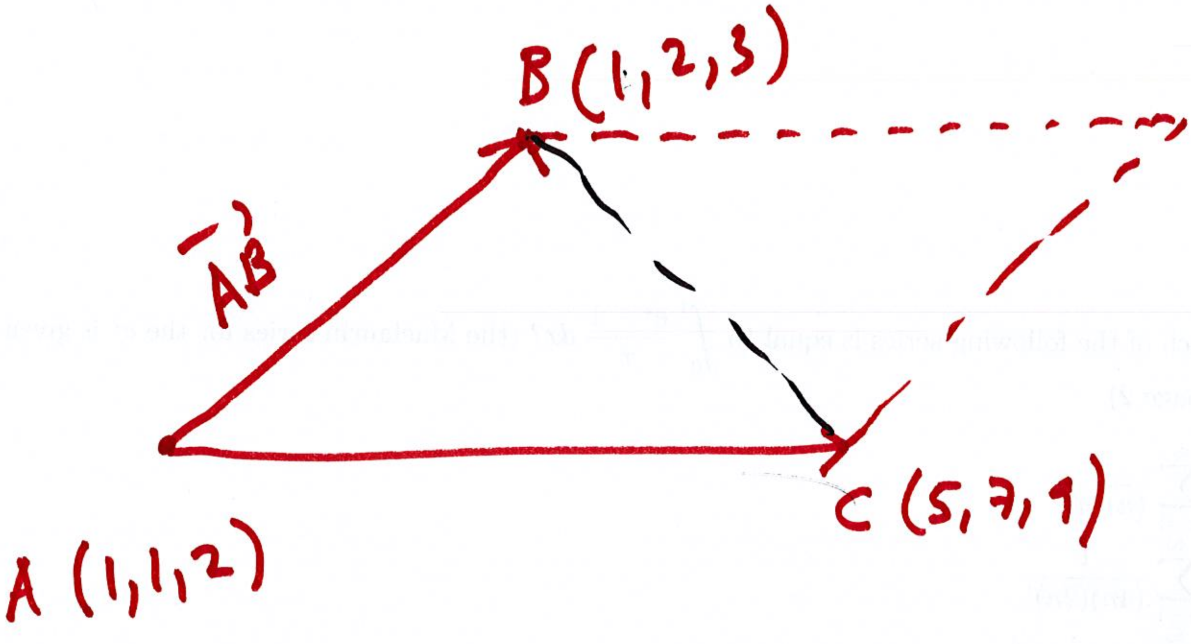
$$= \bar{i}\,(0-(-3)) - \bar{j}\,(0-3)$$

$$+ \bar{k}\,(-1-1)$$

$$= 3\bar{i} + 3\bar{j} - 2\vec{k}\,..$$

Applications   Find the area of the following Triangle.





Area of triangle = $\frac{1}{2}$ Area of the parallelogram.

Recall that the area of the parallelogram

$$= |\vec{AB} \times \vec{AC}|.$$

$$\vec{AB} = \langle 0, 1, 1 \rangle = 0\vec{i} + \vec{j} + \vec{k}$$

$$\vec{AC} = \langle 4, 6, 7 \rangle = 4\vec{i} + 6\vec{j} + 7\vec{k}.$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 4 & 6 & 7 \end{vmatrix}$$

$$= \vec{i}(7-6) - \vec{j}(0-4) + \vec{k}(0-4)$$

$$= \vec{i} + 4\vec{j} - 4\vec{k}.$$

Area of the triangle

$$= \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$$

$$= \frac{1}{2}\sqrt{1+16+16} = \frac{1}{2}\sqrt{33}.$$

$$\text{Volume} = |\vec{A} \times \vec{B} \cdot \vec{C}|$$

How you do this

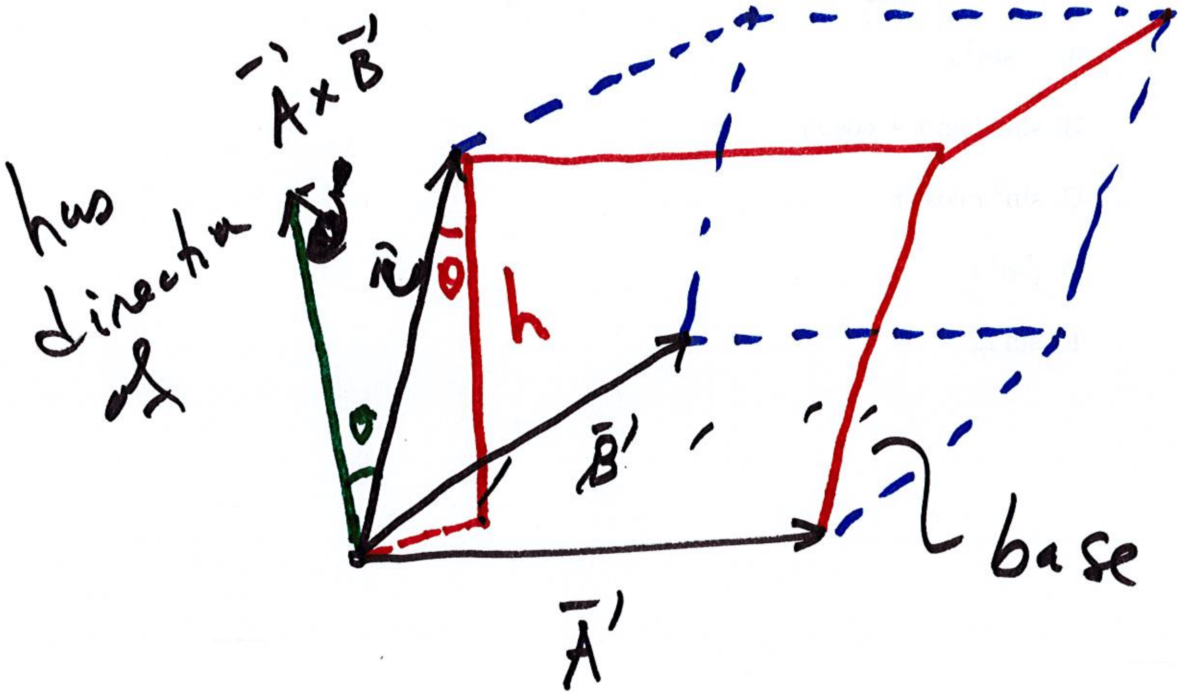
$$\vec{A} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{C} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$$

$$|\vec{A} \times \vec{B} \cdot C| = \left| \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right|$$

PARALLELEPIPED



Volume = Area of base × height

Area of the base = $|\vec{A} \times \vec{B}|$.

h = height = $|\vec{C}| \cdot \cos\theta$.

Volume = $|\vec{A} \times \vec{B}| \cdot |\vec{C}| \cdot \cos\theta$.

θ = Angle between $\vec{C}$ and $\vec{A} \times \vec{B}$.

$$C = c_1 \vec{i} + c_2 \vec{j} + c_3 k.$$

$$\vec{A} \times \vec{B} = \vec{i} \left| \qquad \right| - \vec{j} \left| \qquad \right| + k \left| \qquad \right|$$

$$C \cdot \vec{A} \times \vec{B} = c_1 \left| \qquad \right| - c_2 \left| \qquad \right| + c_3 \left| \qquad \right|$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$