Lesson 8 Sections 6.4 and 6.5.

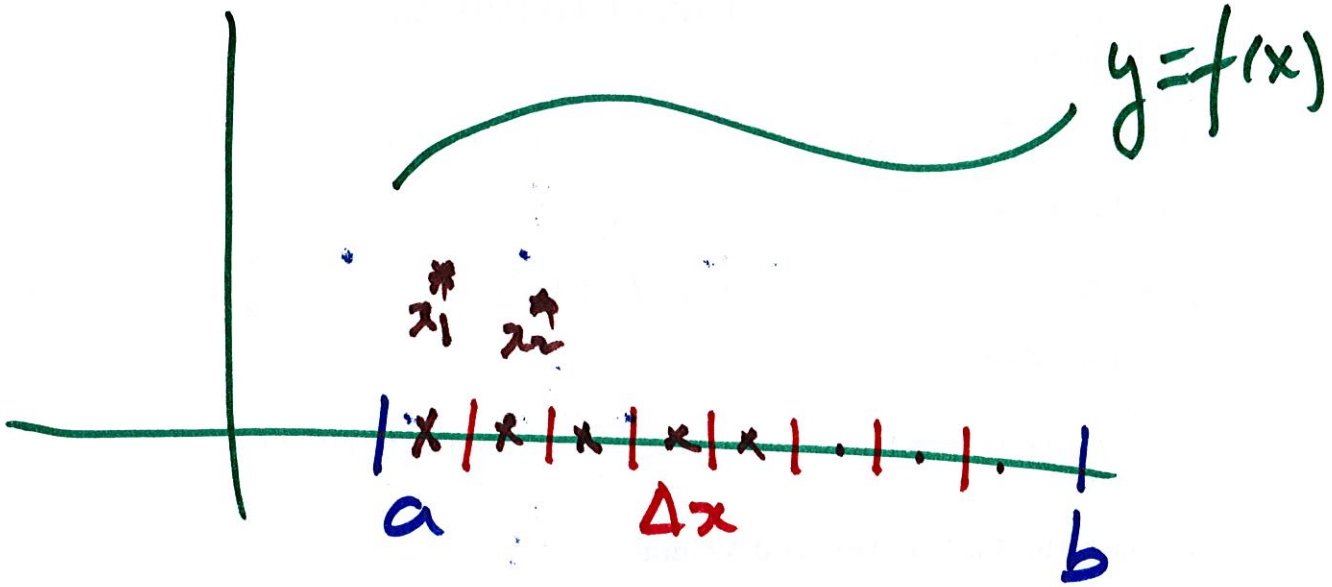
Section 6.5 Average of a function

Given a list of numbers

1, 2, 10, 20, 50, 60

$$\text{Average} = \frac{1+2+10+20+50+60}{6}$$

Generalize this notion to find the average of a function $f(x)$ x in $[a,b]$.



Step 1: Divide the interval $[a, b]$ into n parts (equal)

Each interval $\Delta x = \frac{b-a}{n}$.

Approximate average.

$$\overline{f}_n = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}$$

$$\Delta x = \frac{b-a}{n}$$

So $$n = \frac{b-a}{\Delta x}$$ $\rightarrow \int_a^b f(x)\,dx$

$$\bar{f}_n = \frac{1}{b-a}\left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right]\Delta x$$

$$f_{\text{average}} = f_{\text{ave}} = \lim_{n\to\infty} \bar{f}_n$$

$$\lim_{n\to\infty}\left[f(x_1^*) + \cdots + f(x_n^*)\right]\Delta x$$

$$= \int_a^b f(x)\,dx\,.$$

The Mean Value theorem

1, 2, 3, 5

$$\text{Average} = \frac{1+2+3+5}{4} = \frac{11}{4}$$

Average is not on the list.

If $f(x)$ is continuous on $[a, b]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

There exists c in $[a, b]$ so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example: $f(x) = x^3 + 3$ on $[0, 1]$

find the average of f.

$$f_{ave} = \frac{1}{1} \int_0^1 (x^3 + 3)\,dx$$

$$= \frac{x^4}{4} + 3x \Big|_0^1 = 3 + \frac{1}{4} = \frac{13}{4}.$$

Go back to this example.

$f(x) = x^3 + 3$; $f_{ave} = \frac{13}{4}$.

find c in $[0,1]$ such that

$$f(c) = \frac{13}{4}.$$

$$f(c) = c^3 + 3 = \frac{13}{4}$$

$$c^3 = \frac{13}{4} - 3 = \frac{1}{4}$$

$$c^3 = \frac{1}{4}$$

$$c = \sqrt[3]{\frac{1}{4}} = \frac{1}{\sqrt[3]{4}}.$$

Example: $f(x) = e^x$, $[1, 3]$

$$f_{ave} = \frac{1}{2} \int_1^3 e^x \, dx = \frac{1}{2}(e^3 - e).$$

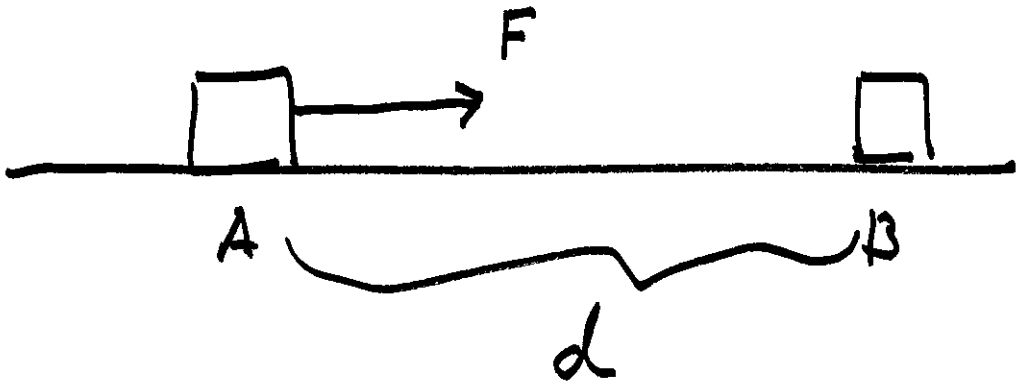
find c in $[1,3]$ such that.

$$f(c) = \frac{1}{2}(e^3 - e)$$

$$e^c = \frac{1}{2}(e^3 - e)$$

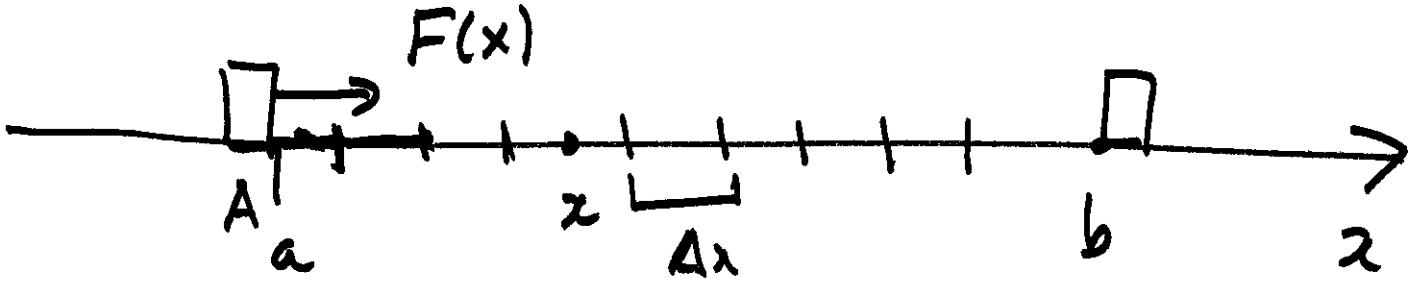
$$c = \ln\left[\frac{1}{2}(e^3 - e)\right]$$

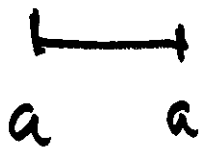
Section 6.5 Work



Work = F.d. Force = Constant.

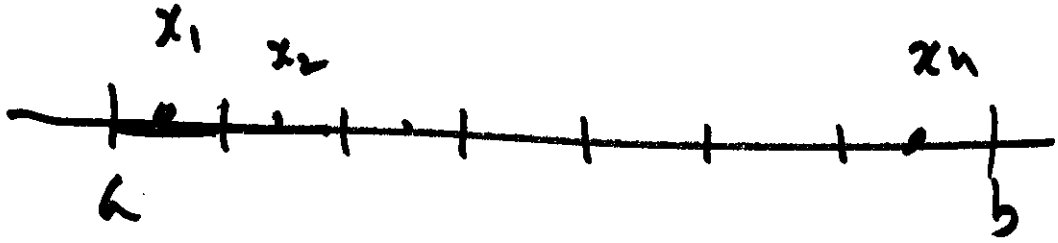
When the force is not a constant.





$F(x^*)\,\Delta x$

Divide $[a,b]$ into n parts



$$W_n = F(x_1)\Delta x + F(x_2)\Delta x + \cdots + F(x_n)\Delta x$$

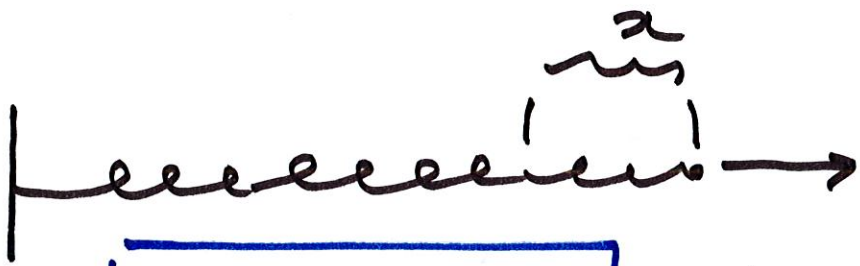
b

$$\text{Work} = \int_a^b F(x)\,dx$$

Example: Hooke's Law

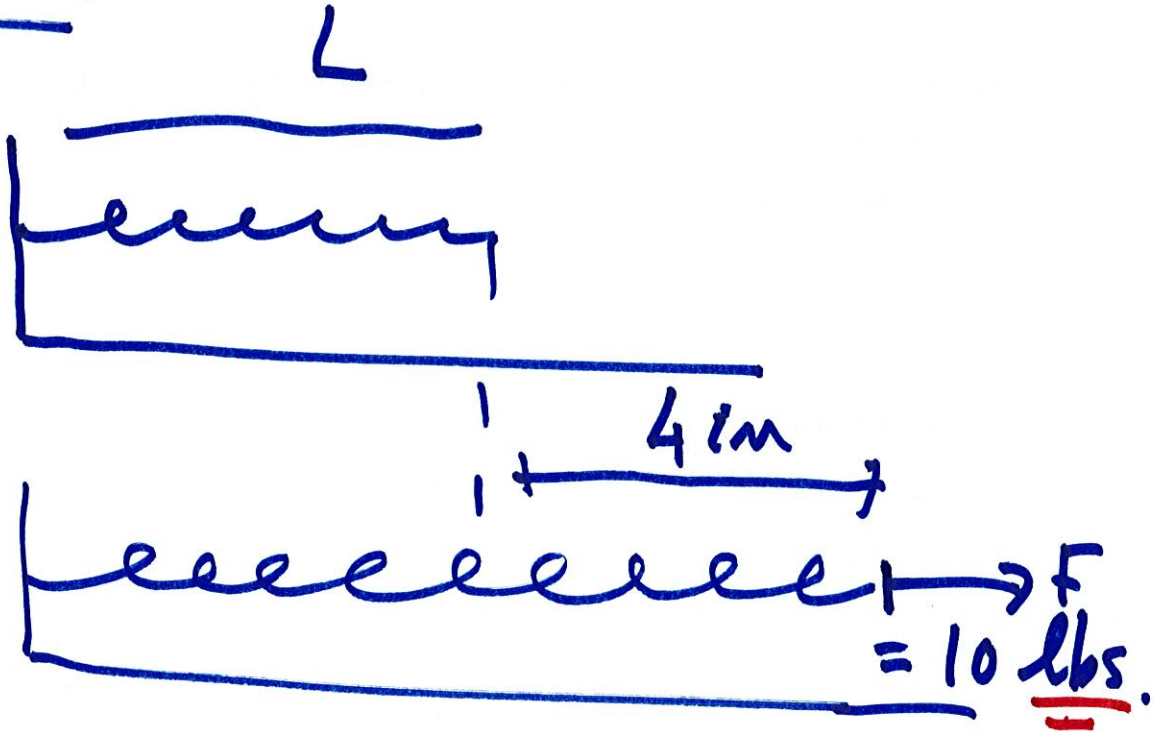
~~Motion~~ Motion of Springs.

L = Natural length of the Spring



$F = kx$. k constant.

Example..



How much work is done to stretch it from its natural length to 6 in. beyond its natural length. $4\text{ in} = \frac{1}{3}\text{ ft}$

$x = 1/3$

Solution: $F = kx$.

$10 = k \cdot 1/3$; $\boxed{k = 30}$

$$W = \int_a^b F(x)\,dx$$

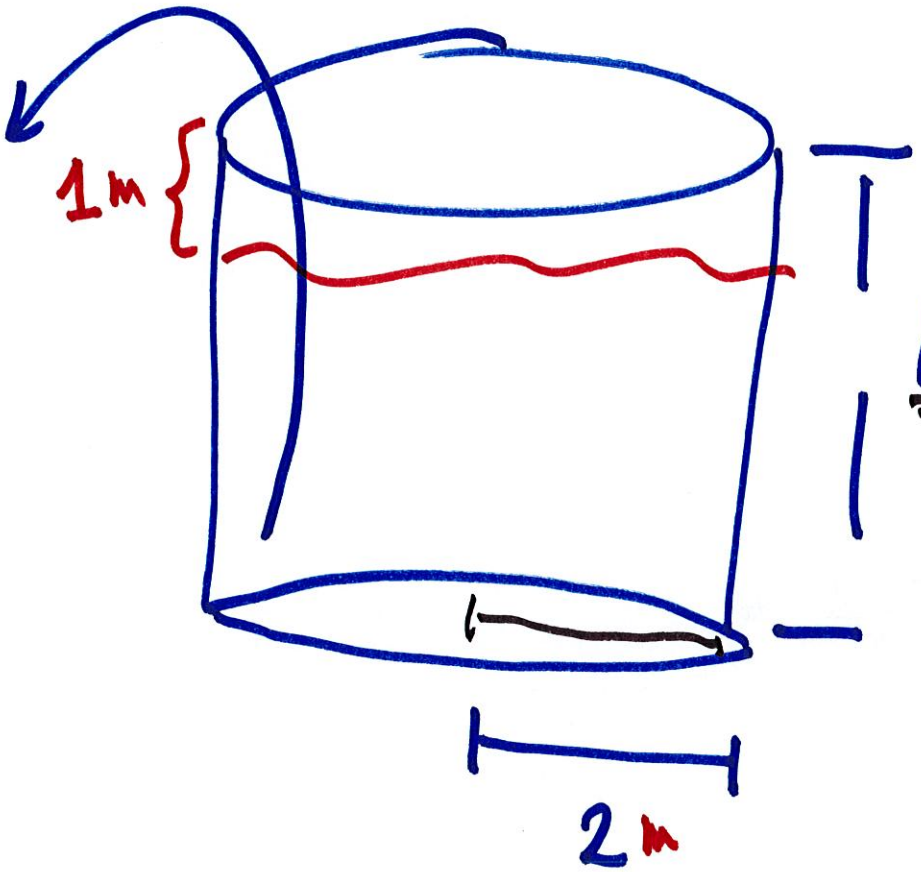
$$F(x) = 30x$$

$$W = \int_0^{1/2} 30x\,dx$$

$$= 30 \cdot \frac{x^2}{2}\Big|_0^{1/2} = \frac{15}{4} \text{ ft-lbs}$$

Work to empty a tank

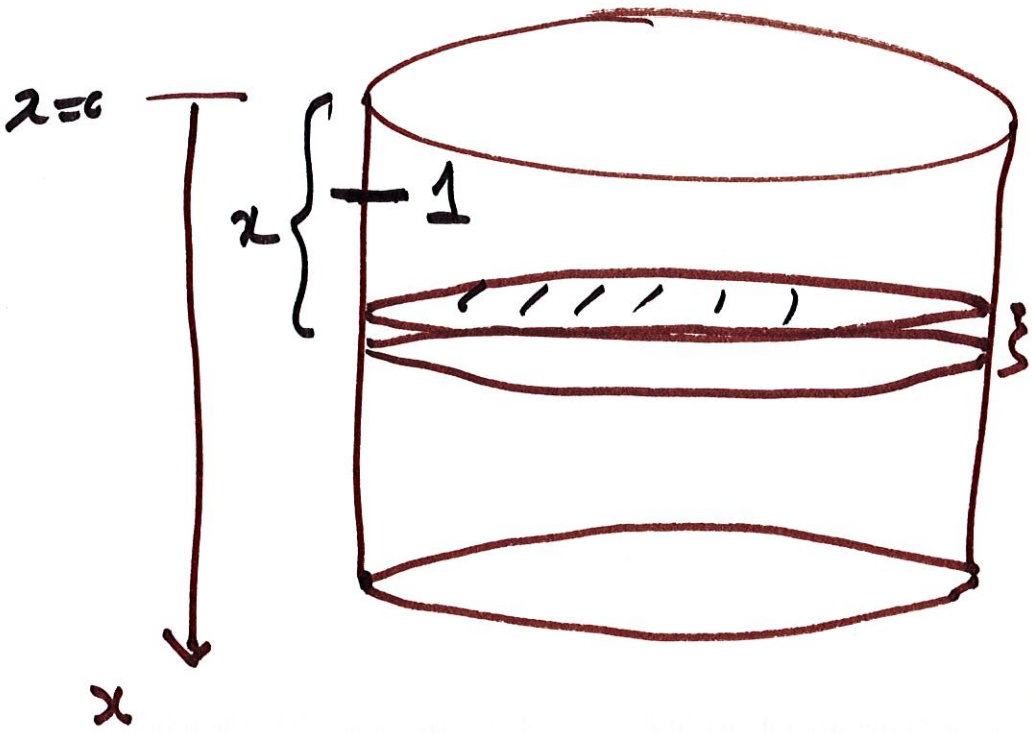
Pump the liquid out.



The tank is full of a liquid of density ρ kg/m^3.

$\rho = 5$ kg/m^3.

Strategy



Volume of slab =
$\pi \cdot (2)^2 \cdot \Delta x$
$= 4\pi \Delta x$

Step 1: Take a slab and find its weight.

= mass · g.

mass = ρ · volume = $5 \cdot 4\pi \Delta x$
$= 20\pi \Delta x$.

$$\text{Weight} = 20\pi g\,\Delta x$$

Step 2: Find the work necessary to move the slab to the top of the tank.

$$= 20\pi g\,\underbrace{\Delta x}\; x$$

Step 3: Total work

$$< \int_1^{10} 20\pi g\, x\, dx$$