Lesson 9 Section 7.1

Integration By Parts

Integration by Parts is the converse of the product Rule.

Product Rule:

$$(fg)' = f'g + fg'.$$

Example: $(xe^x)' = 1.e^x + xe^x$
$= xe^x + e^x$

$$\int x e^x \, dx \qquad \underline{u = x, \ v' = e^x}$$

We use the formula we just obtained

$$(xe^x)' = xe^x + e^x$$

$$xe^x = (xe^x)' - e^x$$

$$\int x e^x \, dx = \int (xe^x)' dx - \int e^x dx$$

$$= xe^x - e^x + C$$

$$\boxed{\int x e^x \, dx = xe^x - e^x + C}$$

Check my answer:

$$(xe^x - e^x + c)' =$$

$$= e^x + xe^x - e^x = xe^x.$$

Question: How do we do this systematically?

$$(uv)' = u'v + uv'$$

$$u'v = (uv)' - uv'$$

$$\boxed{\int u' \underline{\underline{v}}\, dx = \int (uv)' dx - \int uv' dx}$$

Denote $v' dx = dv$

$$u' dx = du$$

$$\int v \underbrace{u' dx}_{du} = uv - \int u \underbrace{v' dx}_{dv}$$

$$\boxed{\int v \, du = uv - \int u \, dv}$$

$$\|$$

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Example:

$$\int \underline{\underline{x}}\ \underbrace{\sin x\ dx}$$

$$\boxed{\int \underline{\underline{u}}\ \underbrace{dv} = uv - \int v\,du}$$

Choose who u and v are.

$u = x,\qquad dv = \sin x\ dx$

$du = dx\qquad v = \int \sin x\ dx = -\cos x$

$$\int \underbrace{x}_{u}\ \underbrace{\sin x\ dx}_{dv} = -x\cos x - \int (-\cos x)\,dx$$

$$\int \underline{\underline{x}} \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

<u>Check my answer:</u>

$$(-x\cos x + \sin x + C)'$$

$$= -\cancel{\cos x} + x\sin x + \cancel{\cos x} = x\sin x$$

Curved ball:

$$\int \sin(\sqrt{x})\, dx.$$

Step 1: Make a substitution

$$w = \sqrt{x} = x^{1/2}$$

$$dw = \frac{1}{2} \underbrace{x^{-1/2}} dx$$

$$dw = \frac{1}{2} \frac{1}{w} dx$$

$$dx = 2w\, dw.$$

$$w = \sqrt{x}$$

$$\int \underbrace{\sin \sqrt{x}}\, dx =$$

$$= \int 2w \sin w \cdot dw$$

$$= 2 \int w \sin w \, dw$$

$$= -2 w \cos w + 2 \sin w + C.$$

$$\boxed{\int \sin \sqrt{x}\, dx = -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + c.}$$

$$\int \underbrace{\ln x}_{u} \; \underbrace{dx}_{dv} = x\ln x - \int x \frac{1}{x}\,dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x.$$

$u = \ln x \quad , \quad dv = dx$

$du = \frac{1}{x}\,dx \qquad \underline{v = x}$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

<u>Conclusion:</u> $\int \ln x\,dx = x\ln x - x + C$

<u>Check my answer:</u> $(x\ln x - x + c)'$

$$= \ln x + x \cdot \frac{1}{x} - 1 = \ln x$$

$$\int x^{10} \ln x \, dx =$$

$$= \int \underbrace{\ln x}_{u} \; \underbrace{x^{10} dx}_{dV}$$

$u = \ln x, \quad dv = x^{10} dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^{11}}{11}$

$$\int x^{10} \ln x \, dx = \frac{x^{11}}{11} \ln x - \int \frac{x^{11}}{11} \frac{dx}{x}$$

$$= \frac{x^{11}}{11} \ln x - \int \frac{x^{10}}{11} dx$$

$$= \frac{x^{11}}{11} \ln x - \frac{x^{11}}{(11)^2} + C.$$

$$\int \underbrace{\tan^{-1}x}_{u} \ \underbrace{dx}_{dv} = uv - \int v\,du$$

$u = \tan^{-1}x$; $dv = dx$

$du = \dfrac{1}{1+x^2}dx$; $v = x$

$$\boxed{\begin{aligned} \tan^{-1}x &= \arctan x \\ (\tan^{-1}x)' &= \frac{1}{1+x^2} \end{aligned}}$$

$$\int \tan^{-1}x \, dx = x \tan^{-1}x$$

$$- \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= - x \tan^{-1}x - \int \frac{x}{1+x^2} \, dx$$

$$\int \frac{x}{1+x^2} \, dx = \int \frac{dw}{2w} = \frac{1}{2} \ln|w| + c$$

Set $w = 1+x^2$, $dw = 2x\,dx$

$$x\,dx = \frac{dw}{2}$$

$$\boxed{\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$$

$$\int \tan^{-1}x \, dx = x \tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

$\cancel{\int \sin}$

$$\int \underbrace{\sin^{-1}x}_{u} \, \underbrace{dx}_{dV} = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

$u = \sin^{-1}x$; $\quad dV = dx$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$; $\quad V = x$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int \frac{dw}{\sqrt{w}}$$

$$w = 1-x^2, \quad dw = -\,2x\,dx$$

$$= -\frac{1}{2}\int w^{-1/2}\, dw = -\frac{1}{2}\cdot 2\, w^{1/2}$$

$$= -\, w^{1/2}.$$

$$\int \frac{x}{\sqrt{1-x^2}} = -\left(1-x^2\right)^{1/2}.$$

$$\int \sin^{-1}x\, dx = x\sin^{-1}x + \left(1-x^2\right)^{1/2} + C.$$

$$\int \underbrace{x^2}_{u} \underbrace{\cos x \, dx}_{dv}$$

$u = x^2, \quad dv = \cos x \, dx$

$du = 2x \, dx, \quad v = \sin x$

$$\int x^2 \cos x \, dx = \underbrace{x^2 \sin x} - 2 \underbrace{\int x \sin x \, dx}$$

$$\int x \sin x \, dx = -x \cos x + \sin x + c$$

$$\boxed{\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.}$$