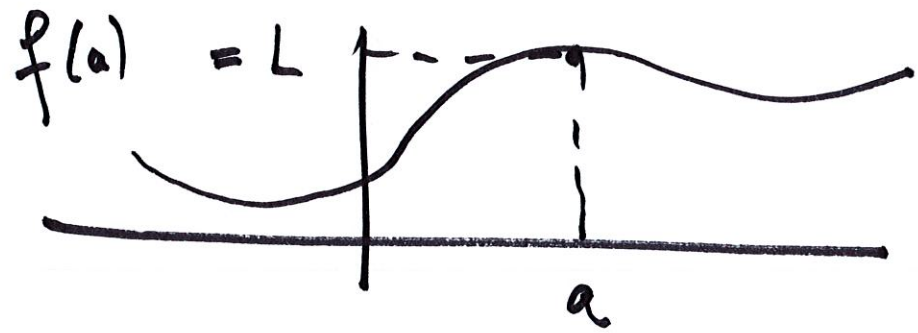
Lesson 10 Section 14.2

Limits and Continuity

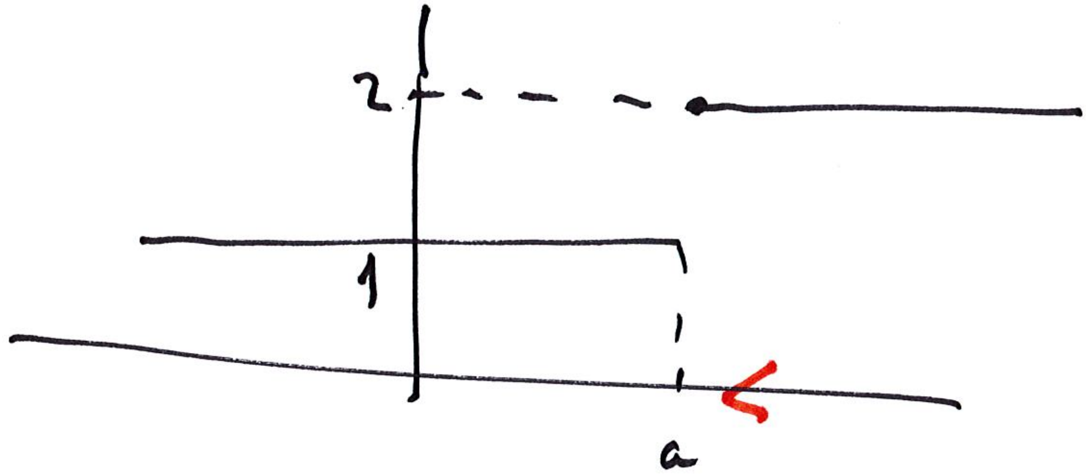
We will discuss

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

1. Dimension $f(x)$.



$$\lim_{x\to a} f(x) = L = f(a).$$

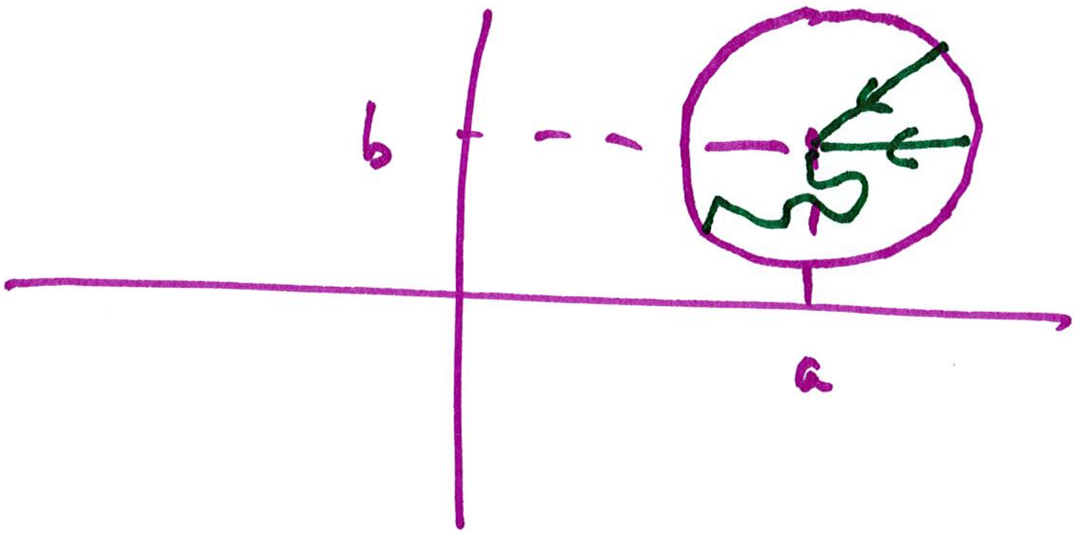


$$f(x) = \begin{cases} 1, & x < a \\ 2, & x \geq a \end{cases}$$

$$\lim_{x \to a^+} f(x) = 2$$

$$\lim_{x \to a^-} f(x) = 1$$

In this case we say the limit does not exist.

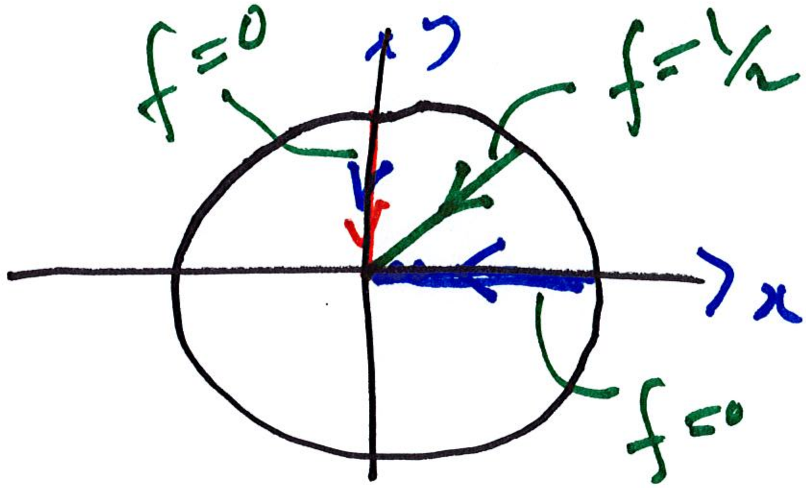


$f(x,y)$ defined near (a,b)

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

Examples: $f(x,y) = \dfrac{xy}{x^2+y^2}$

is defined for $(x,y) \neq 0$.



$$f(x,0) = 0 , \quad x \neq 0$$

$$\lim_{x \to 0} f(x,0) = 0$$

$$f(0,y) = 0 , \quad y \neq 0$$

$$\lim_{y \to 0} f(0,y) = 0$$

If we take $x = y$, $x \neq 0$

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2}$$

$$f(x,x) = \frac{1}{2}$$

Conclusion:

$\lim_{(x,y)\to 0} f(x,y)$ does not exist

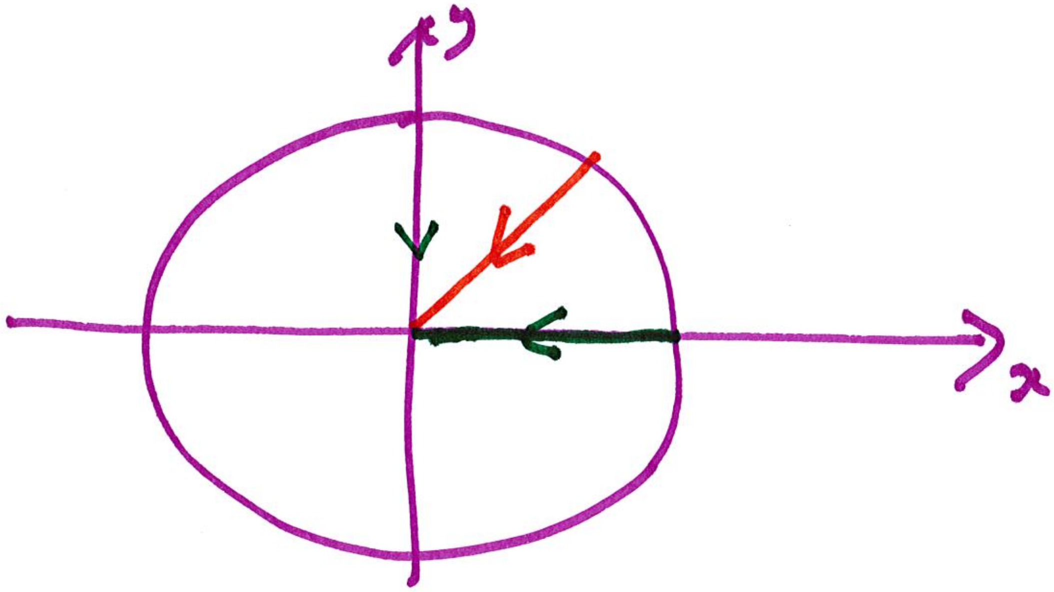
because we get different answers if we take the limit along different paths.

$$f(x,y) = \frac{x\, y^4}{x^2 + y^8}$$

this function is well defined if $(x,y) \neq 0$

Analyze $\lim\limits_{(x,y) \to (0,0)} f(x,y)$

as $(x,y) \to (0,0)$



$y = 0$, $x \neq 0$: $\quad f(x,0) = 0$

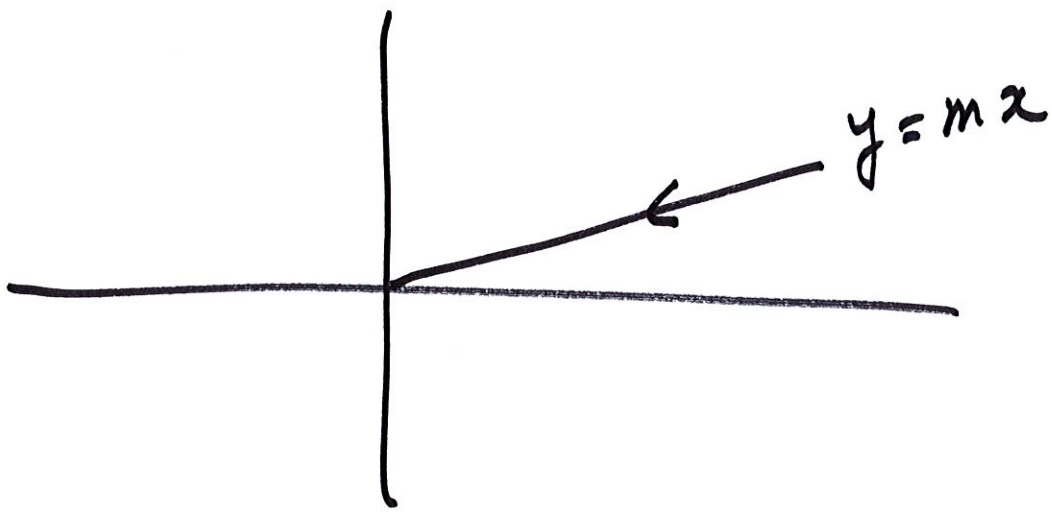
$\underline{x = y}$: $\quad f(x,x) = \dfrac{x^5}{x^2 + x^8}$

$\underline{x \neq 0}$ $\quad f(x,x) = \dfrac{x^2\, x^3}{x^2(1 + x^6)}$

$$f(x,x) = \frac{x^3}{1 + x^6}.$$

$$\lim_{(x,x)\to 0} f(x,x) = 0.$$

Pick any line $y = mx$



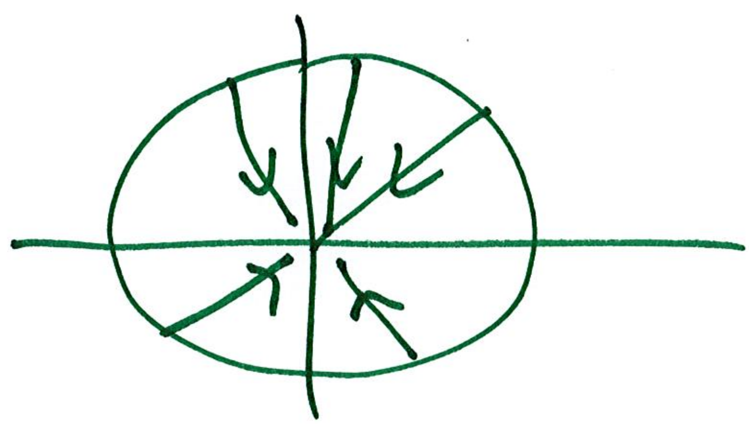
$y = mx \,,\; x \neq 0$

$$f(x, mx) = \frac{x \cdot (mx)^4}{x^2 + (mx)^8}$$

$$f(x, mx) = \frac{m^4 \quad x^5}{x^2 + m^8 \quad x^8} \qquad x \neq 0$$

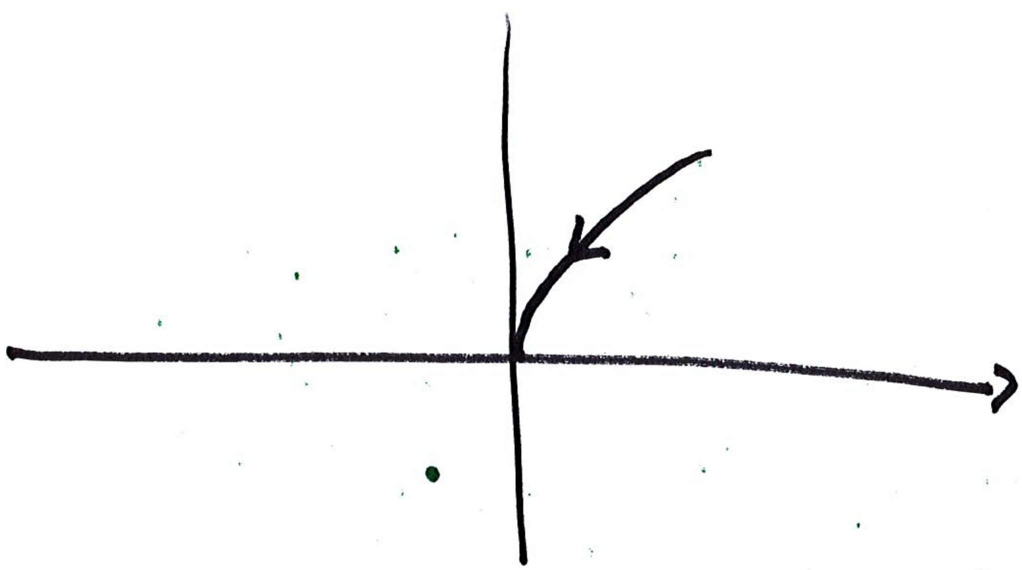
$$f(x, mx) = \frac{m^4 \quad x^3}{1 + m^8 x^6}.$$

$$\boxed{\lim_{x \to 0} f(x, mx) = 0}$$



$$f(x,y) = \frac{x y^4}{x^2 + y^8}$$

$$x = y^4.$$



$$f(y^4, y) = \frac{y^4 \, y^4}{(y^4)^2 + y^8}$$

$$= \frac{y^8}{2y^8} = \frac{1}{2}.$$

Conclusion: $\lim_{(x,y)\to(0,0)} \frac{x y^4}{x^2 + y^8}$ does not exist.

We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $f(x,y)$ approaches L

when (x,y) <u>approaches</u> (a,b)

in any possible way.

<u>Example:</u> $\lim_{(x,y)\to 0} \dfrac{x^4 - y^4}{x^2+y^2}$

Notice that

$$x^4 - y^4 = (x^2+y^2)(x^2-y^2)$$

$(x,y) \neq (0,0)$

$$\frac{x^4-y^4}{x^2+y^2} = \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{x^2+y^2}}$$

$$\frac{x^4-y^4}{x^2+y^2} = x^2-y^2$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2)$$

$$= 0.$$

$$\lim_{(x,y)\to 0} \frac{x y^3}{x^2+y^2}$$

At the beginning we saw that

$$\lim_{(x,y)\to 0} \frac{x y}{x^2+y^2}$$

does not exist

Here we have to use inequalities.

$$(|x|-|y|)^2 = |x|^2 + |y|^2 - 2|x|.|y| \geq 0$$

$$\boxed{|x|^2 + |y|^2 \geq 2|x|.|y|}$$

$$\boxed{|x|.|y| \leq \frac{|x|^2+|y|^2}{2}}$$

$$\lim_{(x,y)\to 0} \frac{x\,y^3}{x^2+y^2}$$

$$\left| \frac{x\,y^3}{x^2+y^2} \right| = \frac{|x|.|y|.|y|^2}{x^2+y^2}$$

$$|x|.|y| \leq \frac{1}{2}\left(|x|^2+|y|^2\right)$$

$$= \frac{1}{2}\left(x^2+y^2\right)$$

$$\left| \frac{x\,y^3}{x^2+y^2} \right| \leq \frac{\frac{1}{2}.\cancel{(x^2+y^2)}.|y|^2}{\cancel{x^2+y^2}}$$

Conclusion: $0 \leq \left| \frac{x\,y^3}{x^2+y^2} \right| \leq \frac{1}{2}|y|^2$

Squeeze theorem

$$0 \le \left| \frac{x y^3}{x^2+y^2} \right| \le \frac{1}{2} |y|^2$$

Since $\lim_{y \to 0} \frac{1}{2}|y|^2 = 0$

$$\lim_{(x,y)\to 0} \frac{x y^3}{x^2+y^2} = 0.$$