# Lesson 11 — Section 14.3

# Partial Derivatives

Example

$$f(x,y) = x^2 + e^{xy} + 2y^2$$

The partial derivative with respect to $x$.

$\frac{\partial f}{\partial x}(x,y) = f_x(x,y)$: we differentiate the function in $x$ and pretend that $y$ is fixed

$$f_x(x,y) = 2x + ye^{xy} + 0$$

$$\boxed{f_x(x,y) = 2x + ye^{xy}}$$

$\dfrac{\partial f}{\partial y}(x,y) = f_y(x,y)$ We treat $x$ as a constant and differentiate in $y$.

$$f_y(x,y) = 0 + xe^{xy} + 4y$$

$$\boxed{f_y(x,y) = xe^{xy} + 4y}$$

$f(x,y) = \ln(1+x^2+y^2)$

Find $\underset{x}{f}(x,y)$, $f_y(x,y)$.

$$f_x(x,y) = \frac{1}{1+x^2+y^2} \cdot 2x = \frac{2x}{1+x^2+y^2}.$$

$$f_y(x,y) = \frac{1}{1+x^2+y^2} \cdot 2y = \frac{2y}{1+x^2+y^2}.$$

<u>Higher Derivatives</u>. Second partial derivatives.

$(f_x)_x = f_{xx}$.

$(f_x)_y = f_{xy}$.

In this example:

$$f(x,y) = \ln(1+x^2+y^2)$$

$$f_x(x,y) = \frac{2x}{1+x^2+y^2}.$$

$$(f_x)_x = f_{xx}(x,y) = \frac{2(1+x^2+y^2) - 2x(2x)}{(1+x^2+y^2)^2}$$

$$= \frac{2+2x^2+2y^2-4x^2}{(1+x^2+y^2)^2}$$

$$= \frac{2+2y^2-2x^2}{(1+x^2+y^2)^2}.$$

$$(f_x)_y = f_{xy} =$$

$$f_x(x,y) = \frac{2x}{1+x^2+y^2}$$

$$(f_x)_y = \frac{0.(1+x^2+y^2) - 2x(2y)}{(1+x^2+y^2)^2}$$

$$f_{xy} = \frac{-4xy}{(1+x^2+y^2)^2}.$$

$$f_y(x,y) = \frac{2y}{1+x^2+y^2}.$$

$$(f_y)_x = \frac{-2y(2x)}{(1+x^2+y^2)^2}.$$

$$f_{yx} = \frac{-4xy}{(1+x^2+y^2)^2}.$$

In this example:

$$f_{xy} = f_{yx}.$$

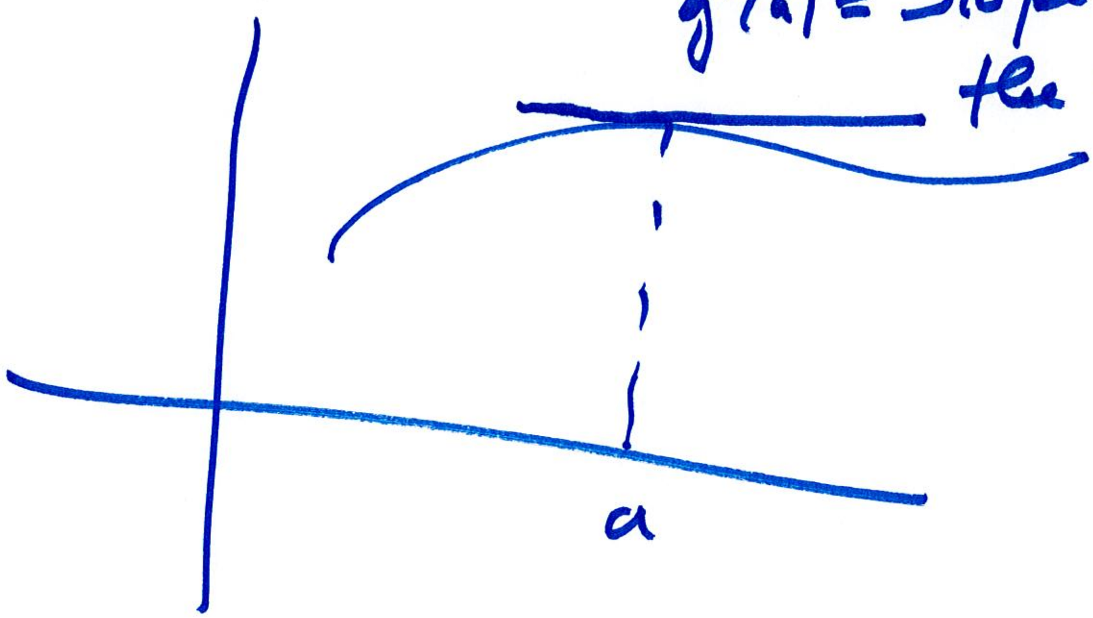
<u>Theorem</u>: If $f_{xy}$ and $f_{yx}$ are continuous on $\mathbb{R}^2$, then

$$f_{xy} = f_{yx}.$$

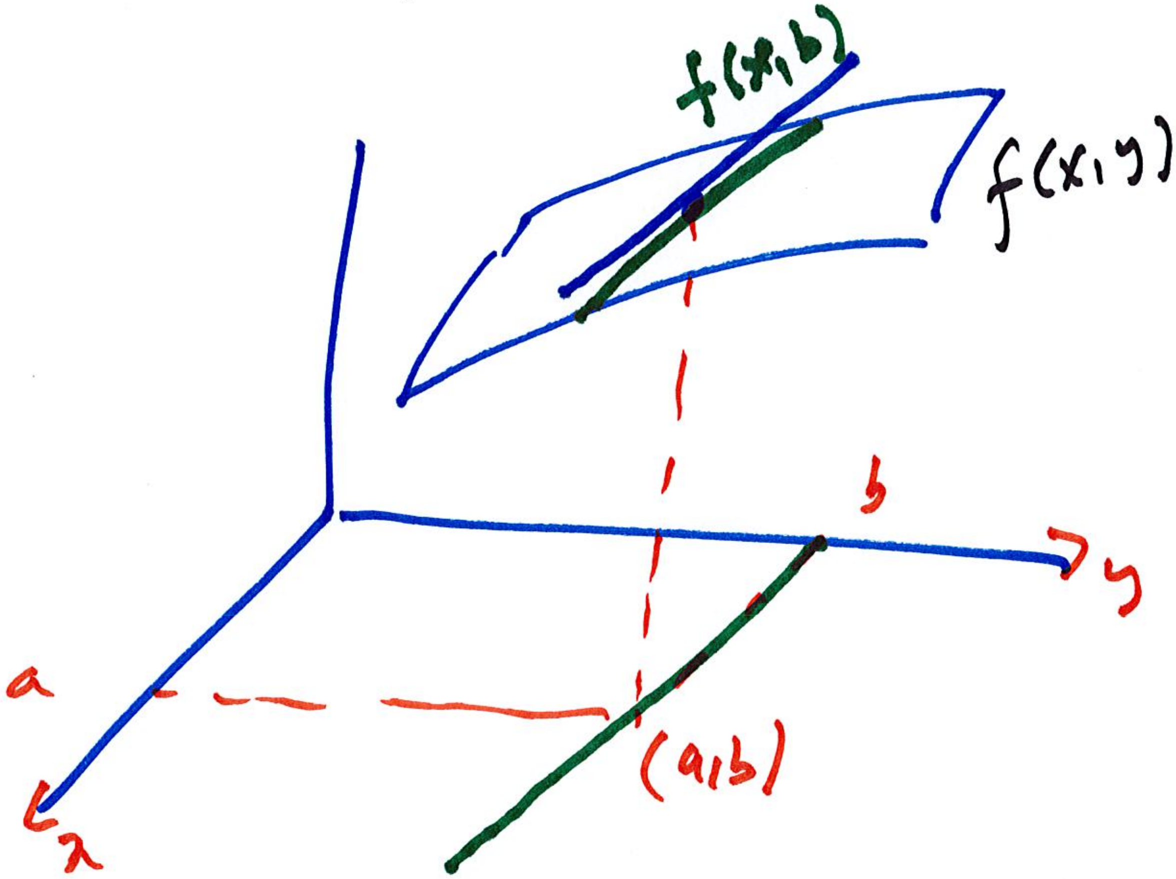
$g(x)$

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

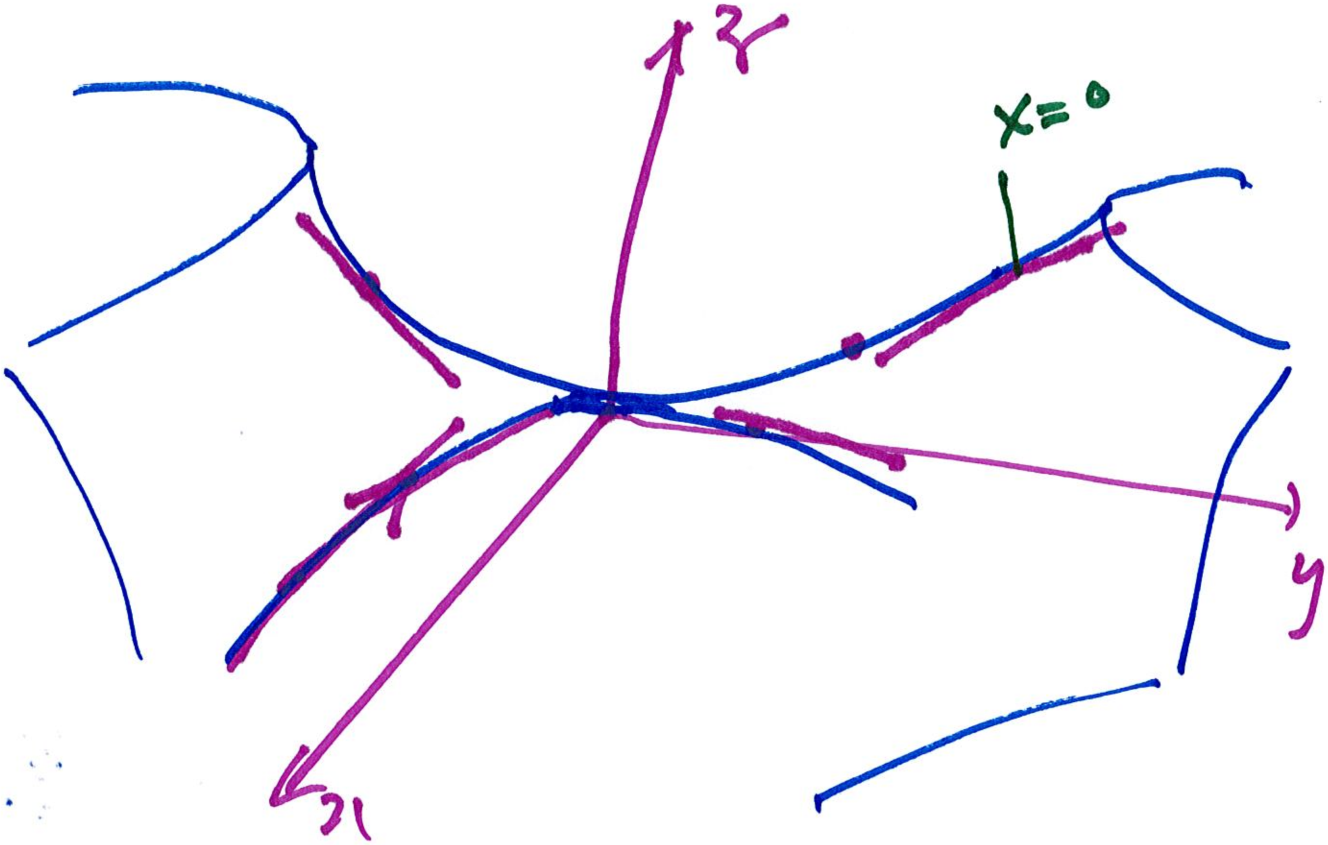
$g'(a)$ = Slope of the tangent.



# The geometric meaning of partial derivatives




$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$




$$z = y^2 - x^2 = f(x,y)$$

$x=0$: $\quad z = y^2 \qquad f_y(0,y) = 2y$

$y=0$: $\quad z = x^2, \qquad f_x(x,y) = -2x$

# Implicit Differentiation

**Example:** $\boxed{z = z(x, y)}$

$$x^2 + y^2 + z^2 + 3xyz = 0.$$

$$z^2 + 3xyz + (x^2 + y^2) = 0$$

$$z = \frac{-3xy \pm \sqrt{9x^2y^2 - 4(x^2+y^2)}}{2}$$

Compute $\frac{\partial z}{\partial x} = z_x$ and $\frac{\partial z}{\partial y} = z_y$

at $\boxed{(1, 1)\ \cancel{\text{and}}\ z = -2}$

$$x^2 + y^2 + z^2 + 3xyz = 0.$$

Take partial derivative in $x$.

$$2x + 2z \cdot z_x + 3yz + 3xy z_x = 0$$

$x = 1$, $y = 1$ and $z = -2$

$$2 + (-4) \cdot z_x + 3(-2) + 3 \cdot z_x = 0$$

$$-4 - z_x = 0$$

$$\boxed{z_x = -4}$$

In general if

$$z = z(x, y)$$

Satisfies

$$F(x, y, z(x,y)) = 0$$

Take partial derivative in $x$

$$F_x(x, y, z(x,y)) + F_z(x, y, z(x,y))\, z_x = 0$$

$$z_x = -\frac{F_x(x,y,z)}{F_z(x,y,z)}$$

Provided $F_z(x, y, z) \neq 0$.

$$z_y = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

if $F_z \neq 0$.