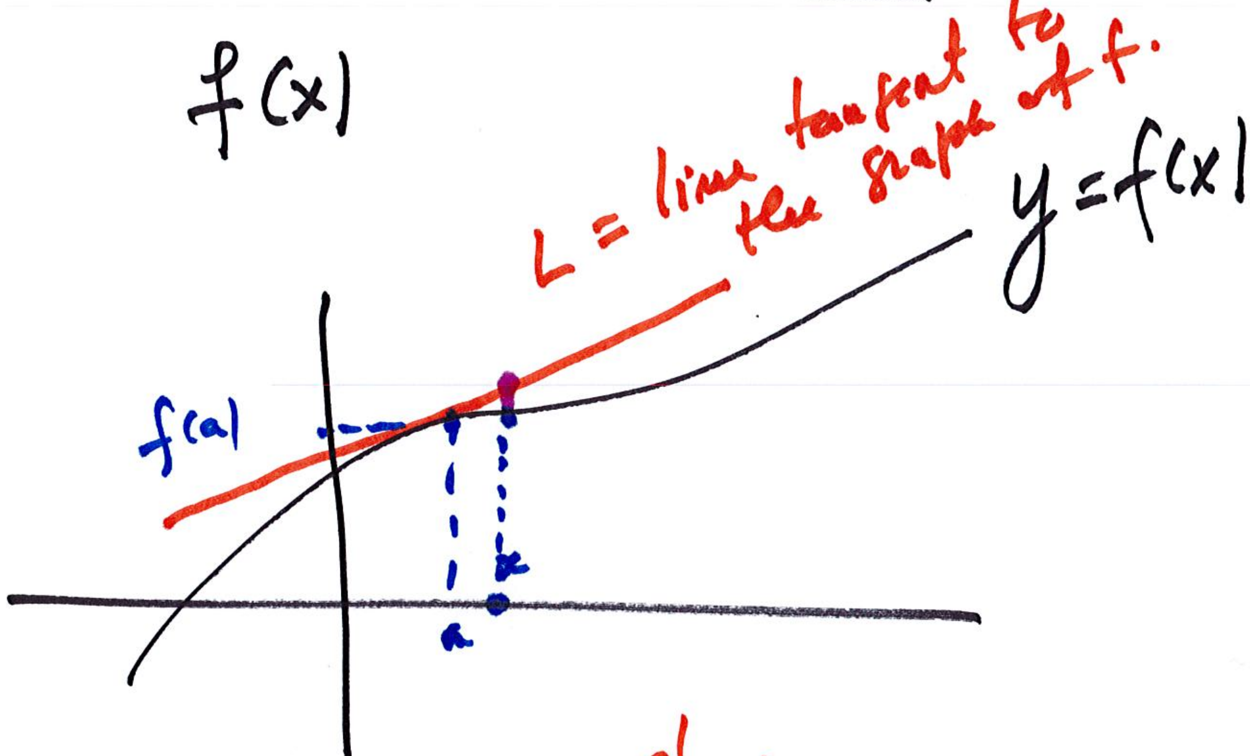
Lesson 12 — Section 14.4

Tangent Planes and Linear Approximation.

The Case of one variable



The slope of L = $f'(a)$

Equation of L:

$$y - f(a) = f'(a)(x-a).$$

Task 1: Find an equation for the plane tangent to $z = f(x,y)$ at the point $(a, b, f(a,b))$.

An equation for this plane has the form

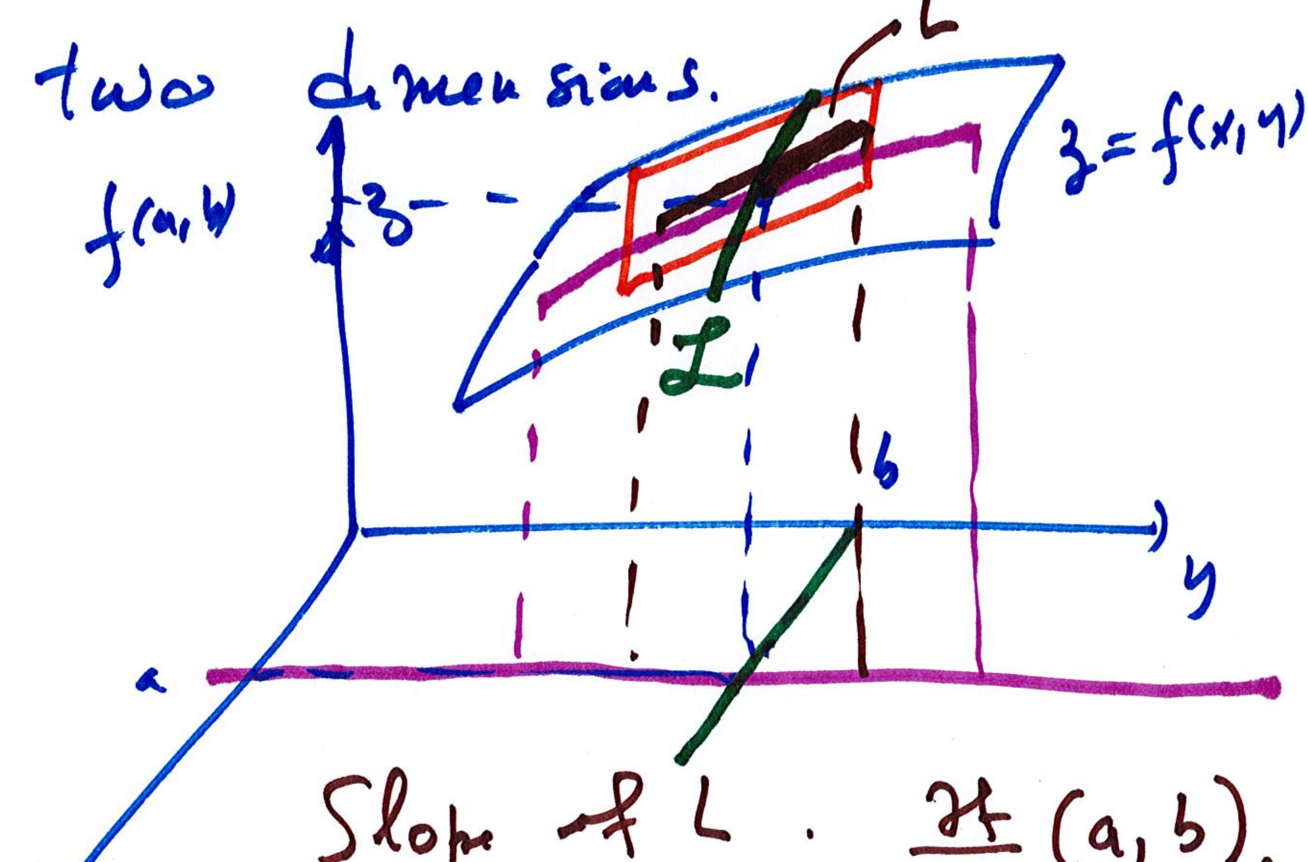
$$z - f(a,b) = A(x-a) + B(y-b)$$

We need to find A and B.

The linear approximation of $f(x)$.

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{\text{Linear approximation}}$$

The analogue of this in two dimensions.



Slope of L : $\dfrac{\partial f}{\partial y}(a,b)$.

Slope of $\mathcal{L}$: $\dfrac{\partial f}{\partial x}(a,b)$

Equation of the Plane

$$z - f(a,b) = A(x-a) + B(y-b)$$

Set $x = a$.

$$z - f(a,b) = \underline{\underline{B}}(y-b)$$

gives an equation for the brown line L.

The slope of this line is

$$= B = \frac{\partial f}{\partial y}(a,b).$$

Set $y = b$

$$z - f(a,b) = A(x-a)$$

$$\text{Slope} = A = \frac{\partial f}{\partial x}(a,b).$$

Conclusion: Tangent plane is given by

$$\boxed{z - \underbrace{f(a,b)} = f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Example: Find an equation for the plane tangent to

$z = x^3 + 4y^2 + 8xy$

at $(1, 1, \underline{\underline{13}})$.

Solution: This is the graph of

$f(x,y) = x^3 + 4y^2 + 8xy$

$f(1,1) = 13$

Here $a=1$, $b=1$

We need to find

$f_x(1,1)$, $f_y(1,1)$.

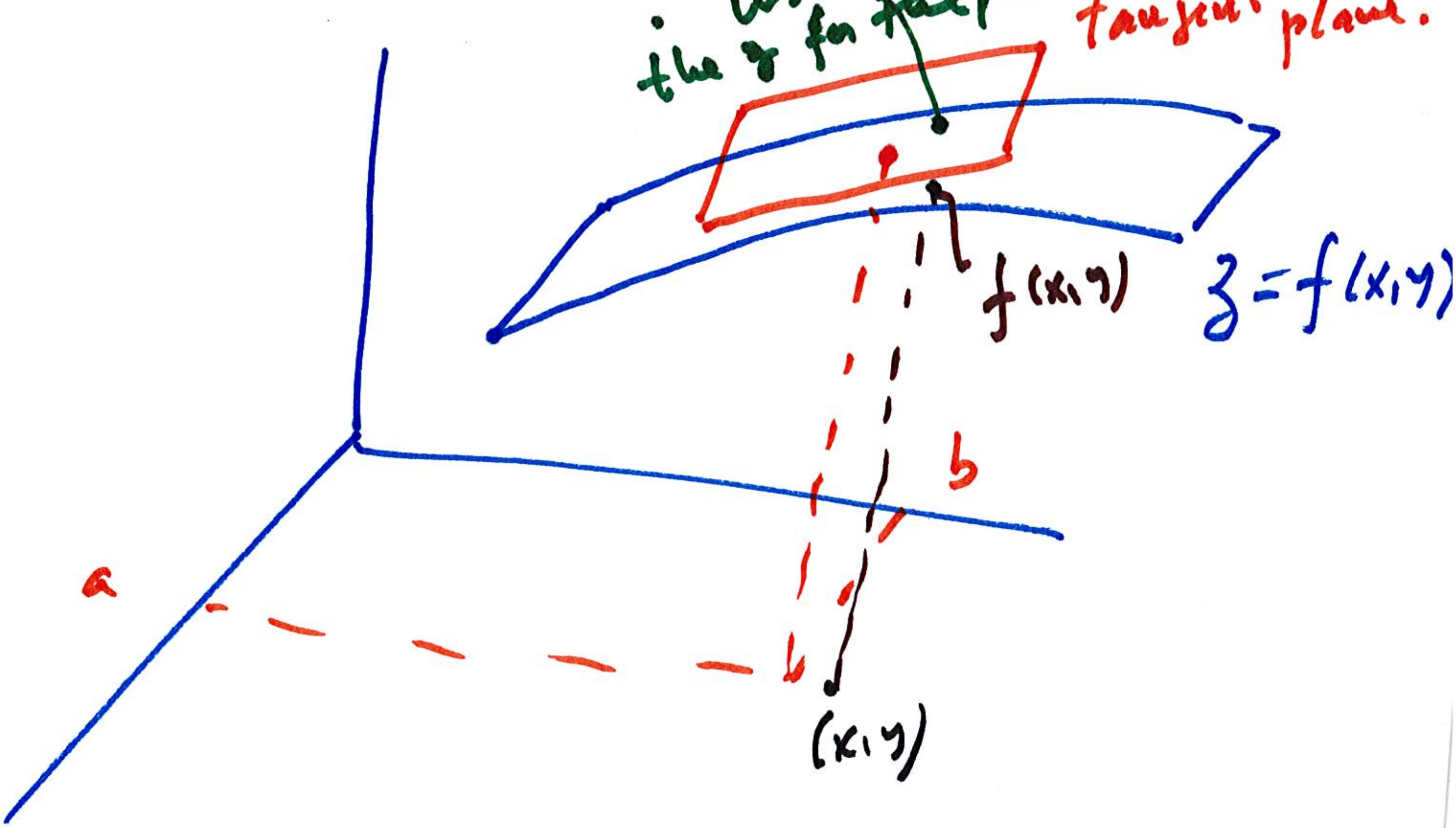
$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 8y =$$

$$f_x(1,1) = 11$$

$$f_y = 8y + 8x \; ; \quad f_y(1,1) = 16.$$

$$z - 13 = 11(x-1) + 16(y-1)$$

$$\boxed{z = 11x + 16y - 14}$$



$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$z = \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}.$$

$$\boxed{f(x,y) \sim f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$$

This is the linear approximation.

Use linear approximation to approximate $\sqrt{(3.2)^2 + (3.9)^2}$.

$$f(x,y) = \sqrt{x^2 + y^2}.$$

$$f(3,4) = \sqrt{9 + 16} = 5.$$

$f(3.2, 3.9) =$ Approximately equal to.

Linear approximation:

$$f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4).$$

$$f_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\boxed{f(3,4) = \frac{3}{5}}$$

$$f_y(x,y) = \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\boxed{f_y(3,4) = \frac{4}{5}.}$$

$$f(\underline{\underline{3.2}}, 3.9) \sim 5 + \frac{3}{5}\cdot(3.2-3)$$
$$+ \frac{4}{5}(3.9-4)$$

$$= 5 + \frac{3}{5}\cdot(0.2) + \frac{4}{5}(-0.1)$$

$$= 5 + \frac{3}{5}\cdot\frac{2}{10} - \frac{4}{5}\cdot\frac{1}{10}$$

$$= 5 + \frac{3}{25} - \frac{2}{25} = 5 + \frac{1}{25} = \frac{126}{25}.$$

Δf and df

$$\boxed{f(x,y) - f(a,b) \doteq \Delta f.}$$

<u>df</u>: The differential of f

is the linear approximation of Δf.

$$f(x,y) \sim f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(x,y) = xy\,e^{x+y}$$

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$f_x = y e^{x+y} + xy e^{x+y}$$

$$= e^{x+y}(y + xy) = y e^{x+y}(1+x)$$

$$f_y = x e^{x+y} + xy e^{x+y}$$

$$= e^{x+y}(x + xy) = x e^{x+y}(1+y)$$

$$\boxed{df(x,y) = e^{x+y} y(1+x)\,dx + e^{x+y} x(1+y)\,dy}$$

~~f(x,y)~~

$$\Delta f = f(x,y) - f(a,b) \sim$$

$$f_x(a,b)\underbrace{(x-a)}_{\Delta x = dx} + f_y(a,b)\underbrace{(y-b)}_{\Delta y = dx}$$

$$\boxed{df = f_x(a,b)\,dx + f_y(a,b)\,dy}$$

Is the differential of f at (a,b).

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

is the differential of f at (x,y).

Let $z = x^3 + 2xy^2$

If (x, y) change from

$(\underline{1}, 2)$ to $(\underline{1.1}, 2.1)$

Compute Δz and dz.

Solution

$$\Delta z = z(1.1, 2.1) - z(1, 2)$$

$$= (1.1)^3 + 2.(1.1)(2.1)^2 - 1 - 8$$

$dz =$ Linear approximation to Δz.

$$dz = f_x(1, 2)(1.1 - 1) + f_y(1, 2)(2.1 - 2)$$

$$= f_x(1, 2)(0.1) + f_y(1, 2)(0.1).$$