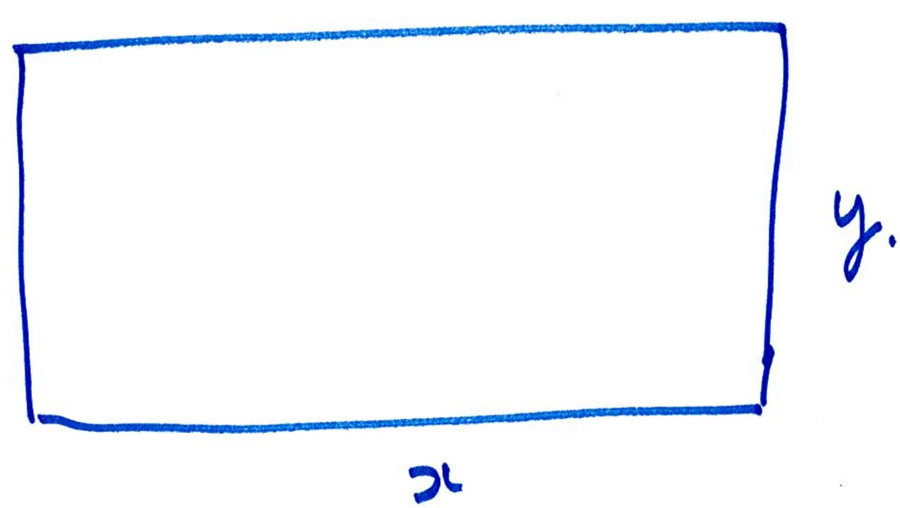
Lesson 13 Sections 14.4 (rest) and 14.5.

Part I: Remainder of 14.4

Examples Use of differentials

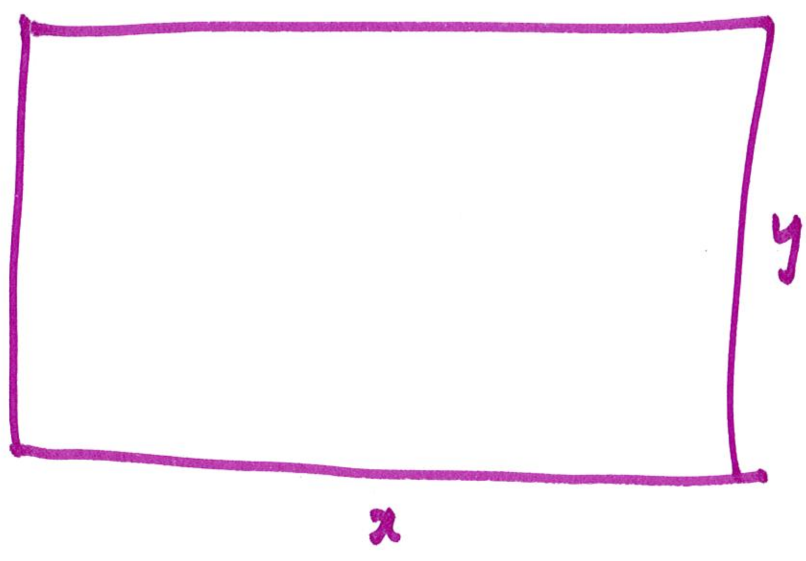
Ex 1: Let E be a rectangle with sides x and y.



Suppose when we measure the sides x and y. We make a 1%

error. Question what would be the maximum error of the measurement of the area of the rectangle.

Solution:



$A = \text{Area} = x\,y.$

$x = \tilde{x} + e$

$\tilde{x} = \text{true value.}$

$y = \tilde{y} + e.$

$$\Delta A = xy - \underbrace{\tilde{x}\tilde{y}}$$

$$\Delta A \sim dA$$

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy$$

$$A = xy$$

$$\frac{1}{100} \frac{dA}{A} =$$ percentage of the error in measurement

= Error in measurement expressed in percentage.

$$\frac{\partial A}{\partial x} = y, \quad \frac{\partial A}{\partial y} = x$$

$$dA = y\,dx + x\,dy$$

$$\frac{dA}{A} = \frac{y\,dx}{xy} + \frac{x\,dy}{xy}$$

$$= \frac{dx}{x} + \frac{dy}{y}$$

$$\frac{dA}{A} = \underbrace{\frac{dx}{x}} + \frac{dy}{y}.$$

$$\frac{1}{100}\ \frac{dx}{x} = \frac{1}{100} = 1\%$$

$$\frac{1}{100}\ \frac{dy}{y} = 1\%.$$

Conclusion: The error in the measurement of the area is approximately 2%.

Functions of 3 Variables

$f(x, y, z)$.

The linear approximation of f at (a, b, c) is

$$f(a,b,c) + \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c).$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Section 14.5: The chain Rule.

The case of one variable

$f(x)$ is differentiable

and $x = g(t)$ is also differentiable, then

$f(g(t))$ is also differentiable

and
$$\boxed{\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)}$$

Example: $\cos(e^t)$

$$\frac{d}{dt}\cos(e^t) = -\sin(e^t).\,e^t$$
$$= -e^t \sin(e^t).$$

The analogue of this for functions of several variables.

Case I: $f(x,y)$ is differentiable.

Suppose that $x = g(t)$ and $y = h(t)$. are differentiable

$$z(t) = f(g(t), h(t))$$

Is also differentiable and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(g(t), h(t)) \cdot \frac{dg}{dt}$$

$$+ \frac{\partial f}{\partial y}(g(t), h(t)) \frac{dh}{dt}.$$

$$\frac{\partial f}{\partial x}(x(t), y(t)) = y(t)\, e^{x(t)\, y(t)}$$

$$= \sin t \; e^{\cos t \sin t}.$$

$$\frac{\partial f}{\partial y}(x(t), y(t)) = \cos t \; e^{\cos t \sin t}.$$

<u>Conclusion:</u>

$$\frac{dz}{dt} = -\sin^2 t \; e^{\cos t \sin t}$$

$$+ \cos^2 t \; e^{\cos t \sin t}$$

$$\boxed{\frac{dz}{dt} = e^{\cos t \sin t}\left(\cos^2 t - \sin^2 t\right)}$$

Example:

$$\text{Let } f(x,y) = e^{xy}$$

$$x(t) = x = \cos t, \quad y = \sin t = y(t)$$

$$z(t) = f(x(t), y(t)).$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}$$

$$+ \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}$$

$$\cancel{\frac{\partial z}{\partial t}} \; \frac{\partial f}{\partial x} = y e^{xy}, \quad \frac{\partial f}{\partial y} = x e^{xy}$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.$$

$$f(x,y) = e^{xy}$$

$x(t) = x = \cos t$, $y = \sin t = y(t)$

$$z(t) = f(x(t), y(t))$$

$$= e^{\cos t \sin t}.$$

$$\frac{dz}{dt} = \ldots$$ Verify you get the same answer.

Example: $f(x,y)$ is such that

$$\frac{\partial f}{\partial x}(1,3)=8, \quad \frac{\partial f}{\partial y}(1,3)=4.$$

Suppose $x = g(t)$

$$g(4)=1, \quad g'(4)=-1$$

$y = h(t)$

$$h(4)=3, \quad h'(4)=-8$$

Find $\frac{d}{dt} z(t)$ at $t=4$

$$z(t) = f(g(t), h(t))$$

$$z(t) = f(g(t), h(t))$$

$$z'(t) = f_x(g(t), h(t)) \cdot g'(t)$$

$$+ f_y(g(t), h(t)) \cdot h'(t)$$

$$z'(4) = f_x(g(4), h(4)) \cdot g'(4)$$

$$+ f_y(g(4), h(4)) \cdot h'(4)$$

$$= f_x(1,3) \cdot (-1) + f_y(1,3) \cdot (-8)$$

$$= 8(-1) + 4(-8) = -8 - 32$$

$$= -40.$$

Case II

$$f(x, y)$$

$$x = x(s,t); \quad y = y(s,t)$$

$$z(s,t) = f\big(x(s,t), y(s,t)\big)$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\big(x(s,t), y(s,t)\big) \cdot \frac{\partial x}{\partial s}$$

$$+ \frac{\partial f}{\partial y}\big(x(s,t), y(s,t)\big) \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\big(x(s,t), y(s,t)\big) \frac{\partial x}{\partial t}$$

$$+ \frac{\partial f}{\partial y}\big(x(s,t), y(s,t)\big) \frac{\partial y}{\partial t}.$$

Example:

$$f(x,y) = x^3 + 6x\,y^3$$

$\rightarrow x = \underbrace{s+t}$, $y = s^2 + t^2$

$$z(s,t) = f(x(s,t),\ y(s,t)).$$

$$\frac{\partial z}{\partial s} = \overbrace{\frac{\partial f}{\partial x}(x(s,t),\ y(s,t))} \cdot \underline{\underline{\frac{\partial x}{\partial s}}}$$

$$+ \frac{\partial f}{\partial y}(x(s,t),\ y(s,t))\,\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial x} = 3x^2 + 6y^3 \ , \quad \frac{\partial f}{\partial y} = 18x\,y^2.$$

$$\frac{\partial x}{\partial s} = 1. \ , \quad \frac{\partial y}{\partial s} = 2s.$$

$$\frac{\partial z}{\partial s} = \left[3(s+t)^2 + 6(s^2+t^2)^3 \right] \cdot$$

$$+ \left[18 \cdot (s+t) \cdot (s^2+t^2)^2 \right] \cdot 2s.$$

In general :

$$f\big(x(t,s,u),\ y(t,s,u),$$
$$z(t,s,u) \big) = W(t,s,u)$$

$$\frac{\partial W}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$