Lesson 15

Continuation of Section 14.6

The gradient vector

$f(x,y)$ differentiable

$\vec{u} = u_1\vec{i} + u_2\vec{j}$, $|\vec{u}| = 1$.

$$D_u f(x,y) = \frac{\partial f}{\partial x}(x,y)\,u_1 + \frac{\partial f}{\partial y}(x,y)\,u_2$$

$$= \vec{\nabla} f(x,y) \cdot \vec{u}.$$

$$\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$f(x,y,z)$ differentiable

$\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$

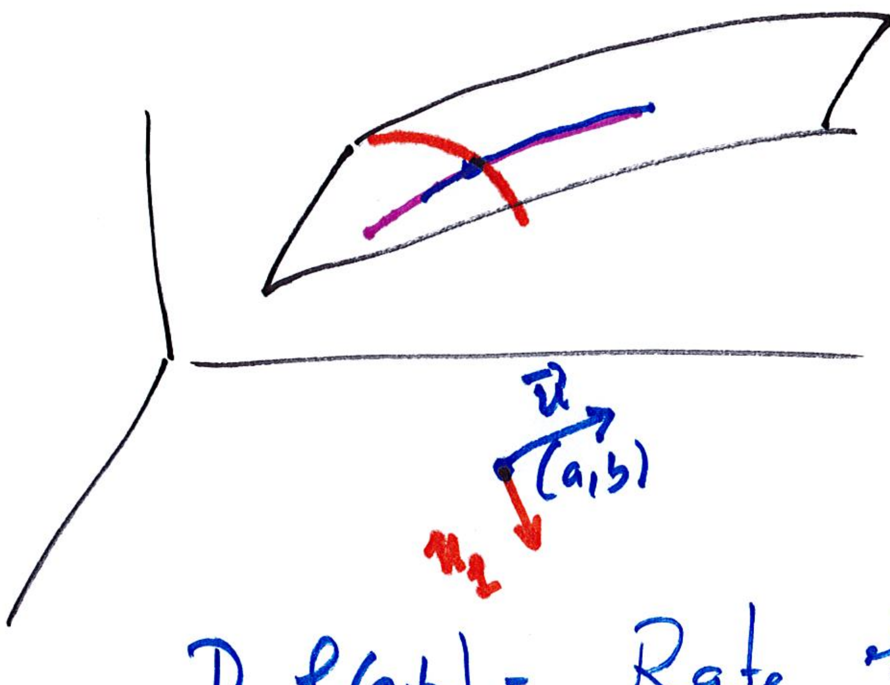
$|\vec{u}| = 1.$

$$D_u f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\, u_1 + \frac{\partial f}{\partial y}(x,y,z)\, u_2 + \frac{\partial f}{\partial z}(x,y,z)\, u_3$$

$$= \vec{\nabla} f(x,y,z) \cdot \vec{u}$$

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

$f'(x)$ = Rate of change of f with respect to x.



$D_u f(a,b)$ = Rate of change of f in the direction of $\vec{u}$?.

Conclusion: $D_u f(a,b)$

is maximal when

$$\vec{u} = \frac{\nabla f(a,b)}{|\vec{\nabla} f(a,b)|}, \quad \nabla f(a,b) \neq 0$$

The direction $\vec{u}$ that maximizes the rate of change of f at (a,b) is

$$\vec{u} = \frac{\vec{\nabla} f(a,b)}{|\vec{\nabla} f(a,b)|}$$

Question: Which vector $\vec{u}$ maximizes $D_u f(a,b)$?

$$D_u f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}$$

$$= |\vec{\nabla} f(a,b)| \cdot |\vec{u}| \cdot \cos\theta$$

$|\vec{u}| = 1.$

$$D_u f(a,b) = |\vec{\nabla} f(a,b)| \cdot \cos\theta$$

$\theta = 0, \quad \cos\theta = 1.$

$$D_u f(a,b) = |\vec{\nabla} f(a,b)| .$$

$$\boxed{\vec{u} = \frac{\vec{\nabla} f(a,b)}{|\vec{\nabla} f(a,b)|}}$$

$$f(x,y) = \sqrt{1-x^2-y^2}.$$

The point in question (a,b) is

such that $f(a,b) = \frac{1}{2} = \sqrt{1-a^2-b^2}$

$$\nabla f(x,y) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2-y^2}}\vec{i}$$

$$-\frac{2y}{2\sqrt{1-x^2-y^2}}\vec{j}.$$

$$\nabla f(x,y) = \frac{-x\,\vec{i}}{\sqrt{1-x^2-y^2}} - \frac{y\,\vec{j}}{\sqrt{1-x^2-y^2}}.$$

$$\nabla f(a,b) = -\frac{a\,\vec{i}}{1/2} - \frac{b\,\vec{j}}{1/2} = -2(a\vec{i}+b\vec{j}).$$

<u>Examples</u>: Let

$$f(x,y) = x + y + \sin(xy)$$

Find the maximum and minimum rates of change of f at $(1, \pi/2)$ and the direction in which these occurs.

<u>Solution</u>: $D_u f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}$

is maximal when $\vec{u} = \dfrac{\nabla f(a,b)}{|\nabla f(a,b)|}$

and in this case

$$D_u f(a,b) = |\vec{\nabla} f(a,b)| .$$

$$f(x,y) = x + y + \sin(xy)$$

$$\frac{\partial f}{\partial x} = 1 + y\cos(xy)$$

$$\frac{\partial f}{\partial y} = 1 + x\cos(xy)$$

$$\frac{\partial f}{\partial x}(1, \pi/2) = 1 + \pi/2 \cos \pi/2 = 1.$$

$$\frac{\partial f}{\partial y}(1, \pi/2) = 1 + \cos \pi/2 = 1.$$

$$\vec{\nabla} f(1, \pi/2) = \frac{\partial f}{\partial x}(1, \pi/2)\,\vec{i} + \frac{\partial f}{\partial y}(1, \pi/2)\,\vec{j}.$$

$$= \vec{i} + \vec{j}.$$

$$\vec{u} = \frac{\nabla f(1, \pi/2)}{|\nabla f(1, \pi/2)|} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j}).$$

Maximum rate of change

$= |\nabla f(1, \pi/2)| = \sqrt{2}.$

How about the minimum?

$$D_u f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}$$

$$= |\vec{\nabla} f(a,b)| \cdot \cos\theta$$

The minimum occurs when

$\cos\theta = -1$, $\theta = \pi$

Minimum occurs when

$\vec{u} = -\frac{\nabla f}{|\nabla f|}$ and is equal to $-|\nabla f|$.

In this case the minimum rate of change is $-\sqrt{2}$

and occurs when

$$\vec{u} = -\frac{1}{\sqrt{2}}(\vec{i} + \vec{j}).$$

2) $f(x,y,z) = x + y + e^{z}$

find the max. rate of change of f at $(1,0,1)$ and the direction in which this occurs.

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= \vec{i} + \vec{j} + e^{z}\vec{k}.$$

$$\nabla f(1,0,1) = \vec{i} + \vec{j} + e\vec{k}.$$

Max rate of change $= |\nabla f(1,0,1)|$

$$= \sqrt{2+e^2}$$

Direction in which this occurs

$$\boxed{\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2+e^2}}(\vec{i} + \vec{j} + e\vec{k})}$$

Find all points (x, y, z)

at which the direction of

fastest rate of change of the

function

$$f(x,y,z) = x^2 + xy - 3e^y + z^3$$

is $\frac{1}{\sqrt{2}}(\vec{i} + \vec{k}) = \vec{u}$.

<u>Solution:</u> The fastest rate

of f is achieved in the

direction

$$\vec{u} = \frac{\vec{\nabla} f(x,y,z)}{|\nabla f(x,y,z)|} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j}).$$

$\nabla f(x, y, z) = (2x+y)\vec{i}$

$+ (x - 3e^y)\vec{j} + 3z^2\vec{k}.$

The $\vec{j}$ component is equal to zero: So

$\rightarrow x - 3e^y = 0$

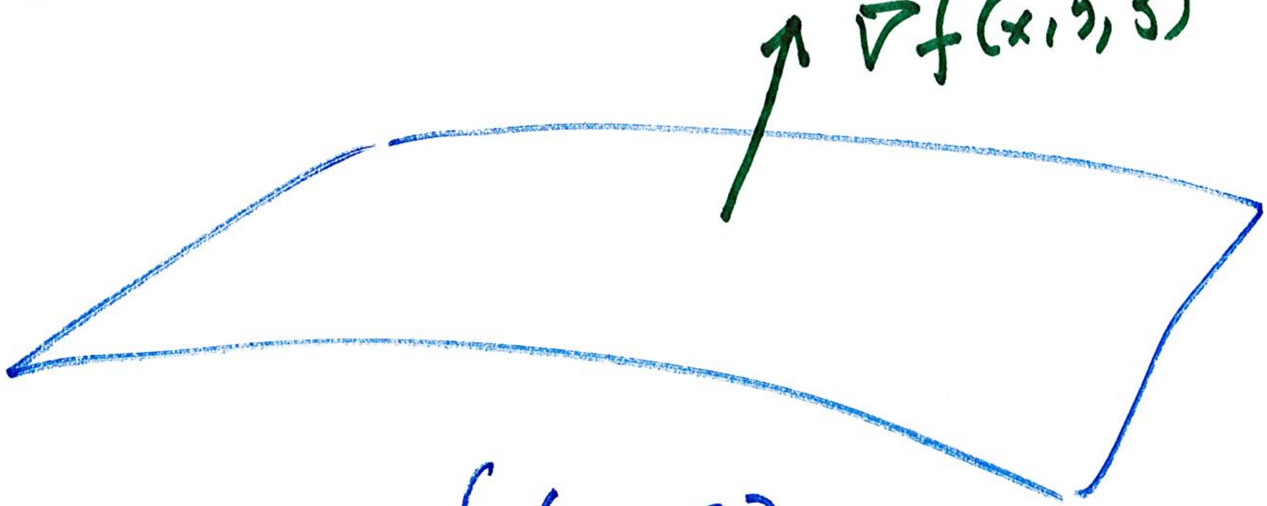
The $\vec{i}$ and $\vec{k}$ components are equal:

$\rightarrow 2x + y = 3z^2.$

So $x = 3e^y$ and. Substituting thus in the next equation

$6e^y + y = 3z^2.$

One last thing



$\nabla f(x,y,z)$ is perpendicular to its level surfaces.