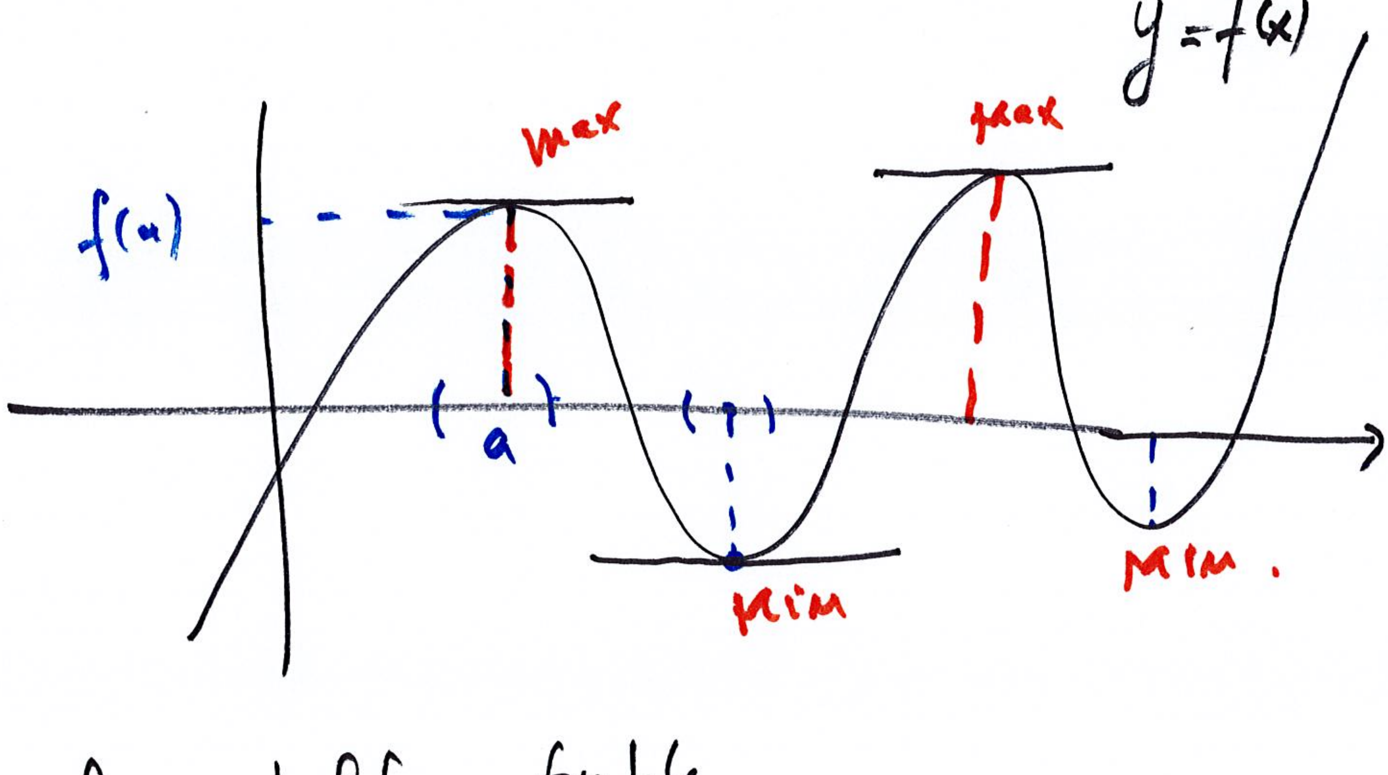
Lesson 16   Section 14.7

Maximum and Minimum values.

Review The 1-D case



f is differentiable.

$f(a)$ is a local max if $f(a) \geq f(x)$ for x near a.

$f(a)$ is a local minimum if

$f(a) \leq f(x)$ for all

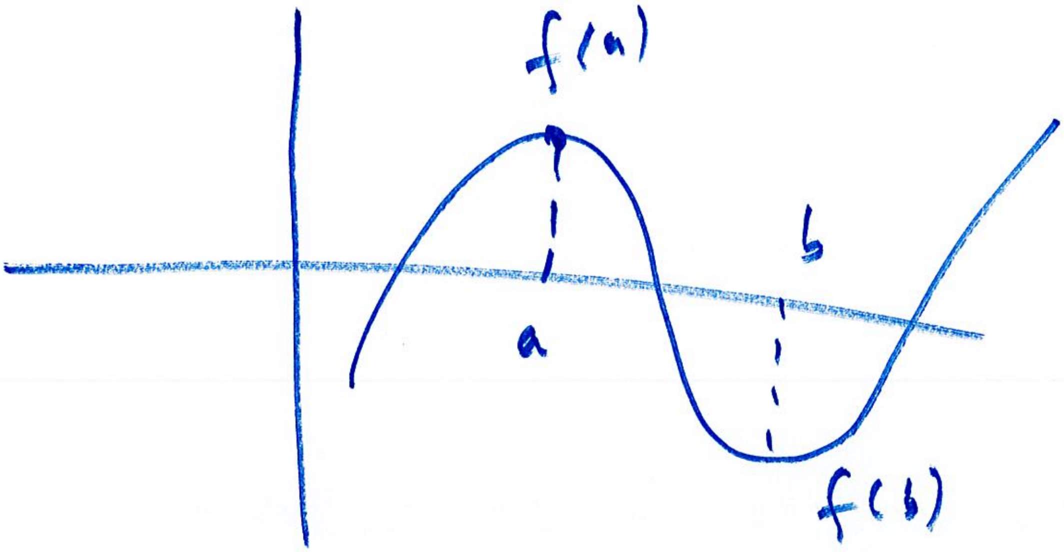
x near a.

Recall that when $f(a)$ is either a local maximum or a local minimum

$$f'(a) = 0$$

<u>Question:</u> How do we distinguish between maxima and minima?

When $f'(a) = 0$, a is called a critical point.



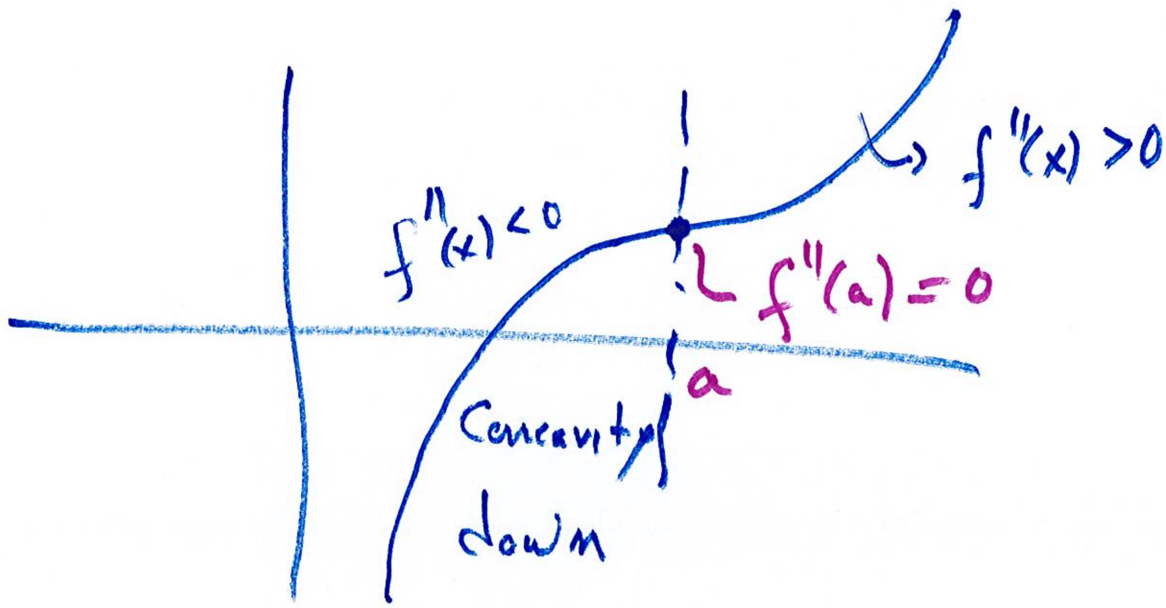
$f''(a) < 0$ whenever $f(a)$ is a local max.

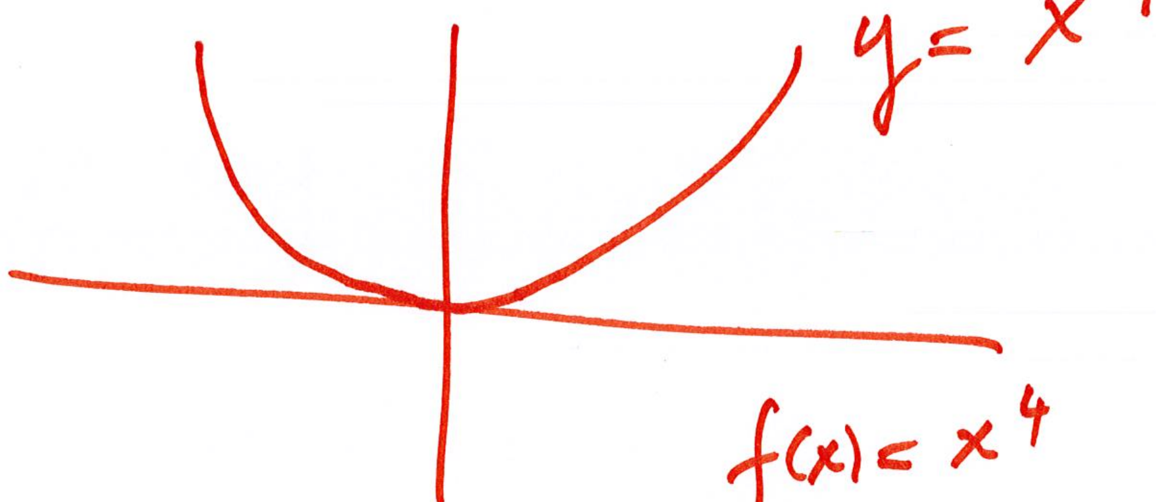
$f''(b) > 0$ whenever $f(b)$ is a local minimum.

Conclusion:

(1) If $f'(a) = 0$ and $f''(a) < 0$ $f(a)$ is a local maximum.

(2) If $f'(a) = 0$ and $f''(a) > 0$ $f(a)$ is a local minimum.





$f(x) = x^4$

$f'(x) = 4x^3$

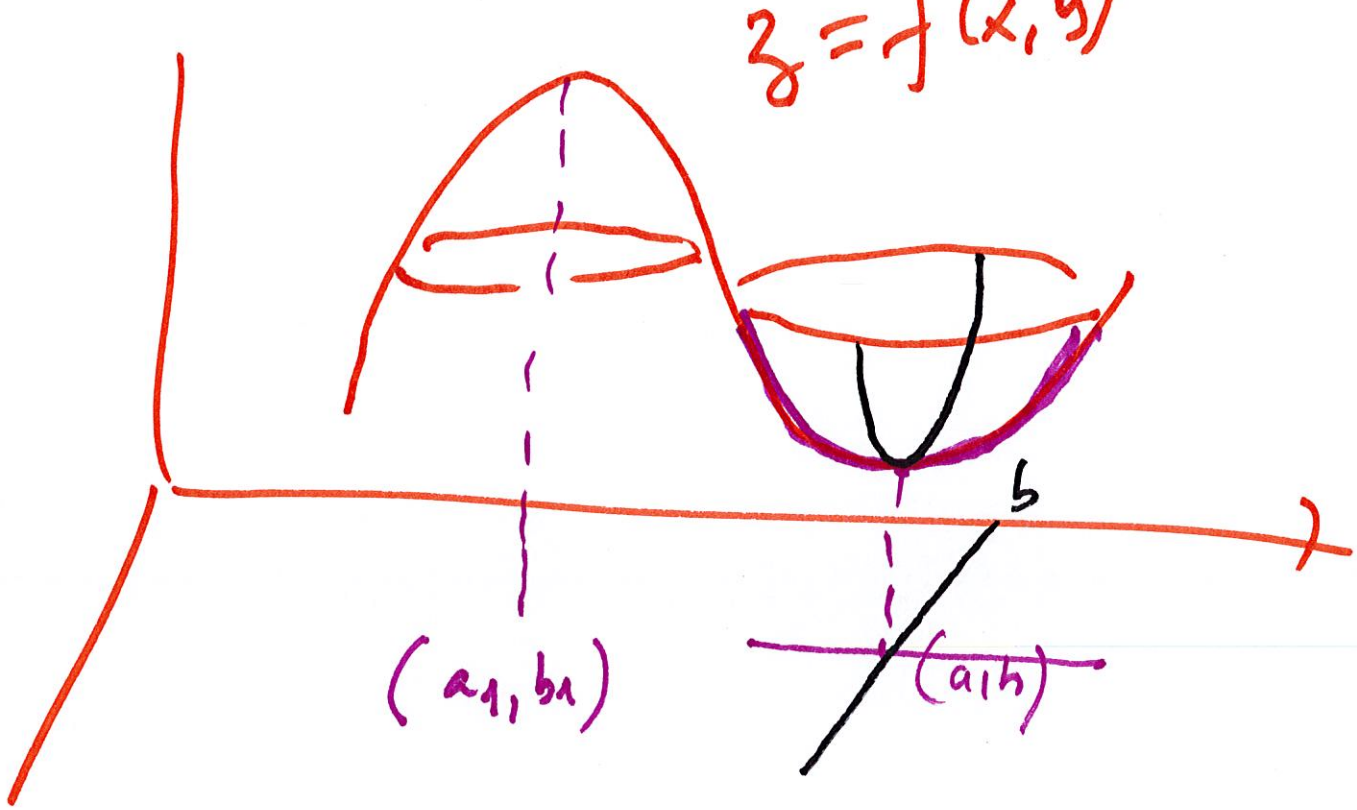
$f''(x) = 12x^2.$

$f''(0) = 0.$

$f(x, y)$ is a differentiable function.

$f(a, b)$ is a local max if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b)

$f(a, b)$ is a local min if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b).



$f(a,b)$ is a local minimum

$f(a_1,b_1)$ is a local maximum.

$$\frac{d}{dx} f(x,b) = 0 \quad \text{when } x = a \qquad \boxed{f_x(a,b) = 0}$$

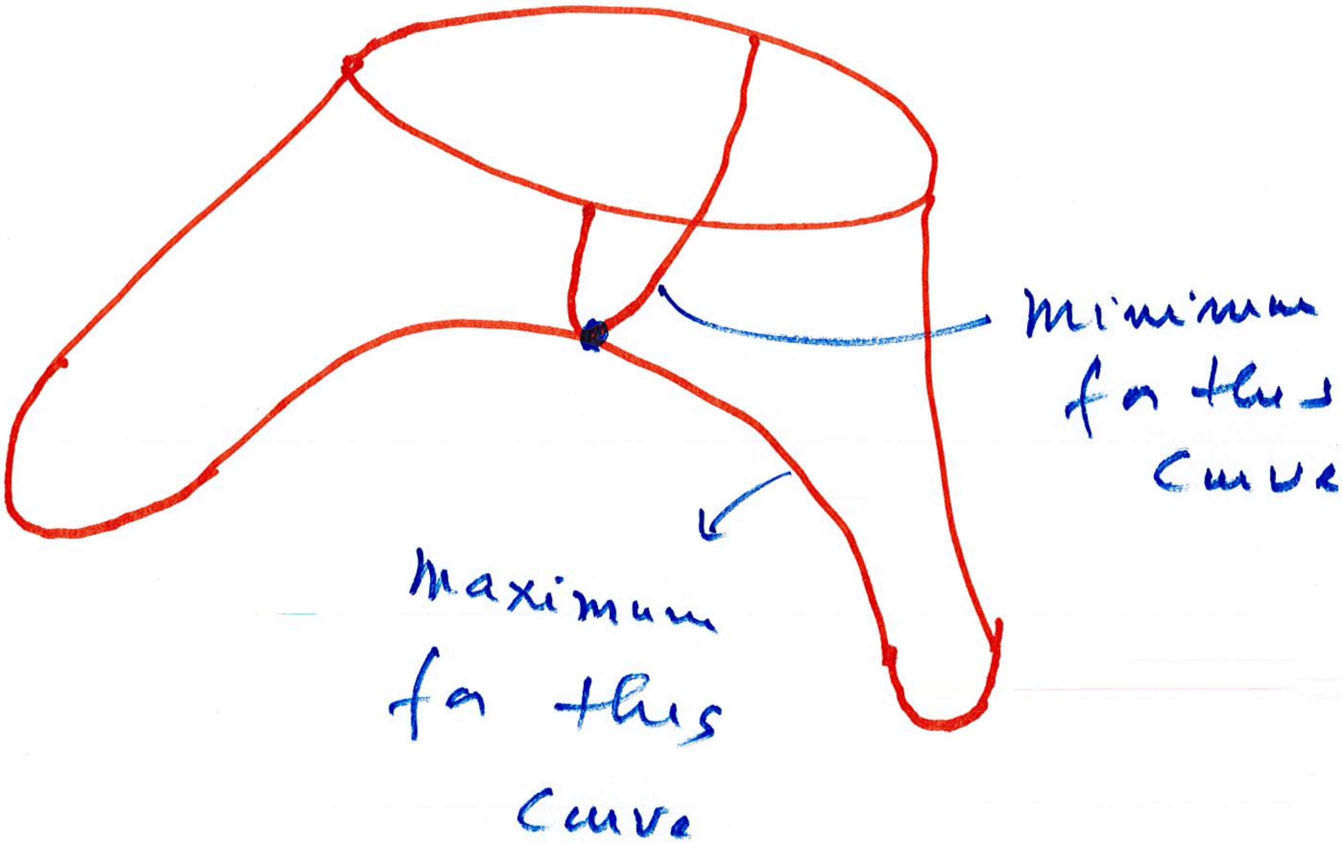
$$\boxed{f_y(a,b) = 0}$$

Whenever $f(a,b)$ is a local maximum or a local minimum

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0.$$

<u>Critical points</u> (x,y) is a critical point of f if

$$f_x(x,y) = 0, \quad f_y(x,y) = 0.$$

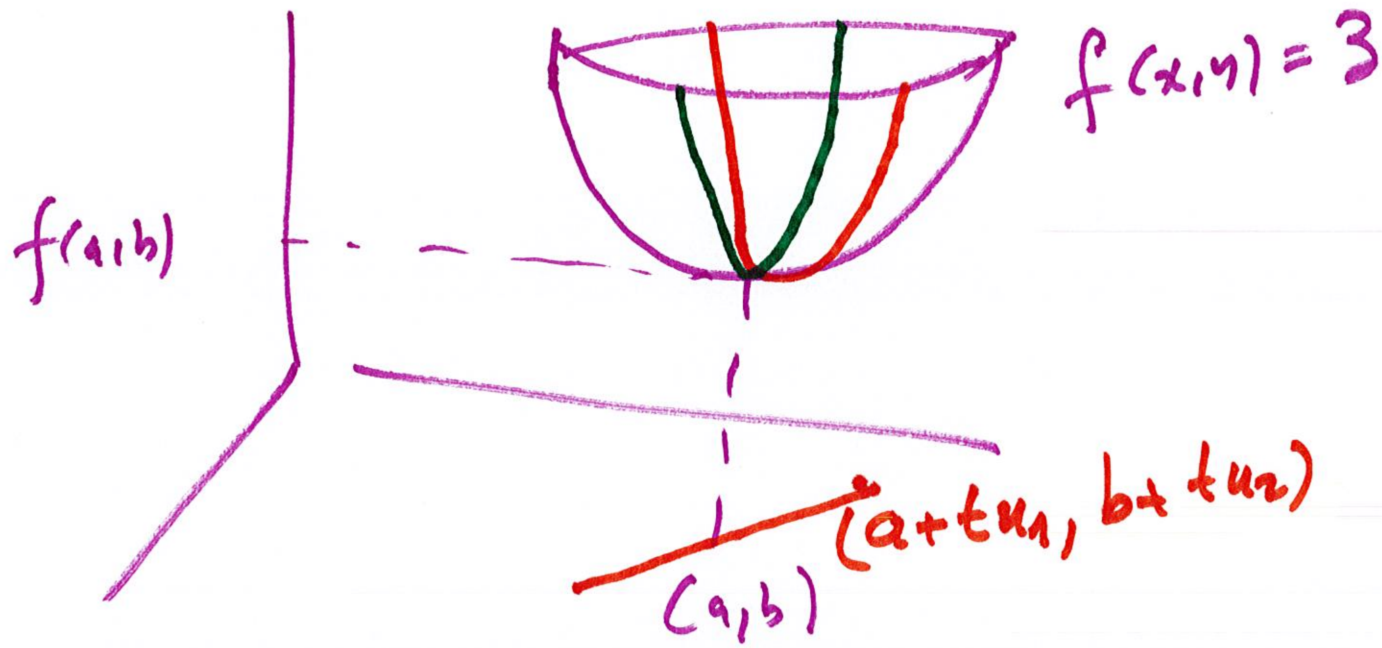


$f_x(a,b) = 0$, $f_y(a,b) = 0$

neither

but $f(a,b)$ is ~~not~~ a max nor a min.

The second derivative test



$$g(t) = f(a+tu_1, b+tu_2)$$

$g(0)$ is a <u>minimum</u> and thus does not depend on (u_1, u_2).

$g'(0)=0, \quad g''(0) > 0.$

$$g'(t) = f_x(a+tu_1, b+tu_2) \cdot u_1$$

$$+ f_y(a+tu_1, b+tu_2) \cdot u_2$$

$$g'(0) = f_x(a,b)\, u_1 + f_y(a,b)\, u_2 = 0$$

for every u_1 and u_2

$$f_x(a,b) = 0, \quad f_y(a,b) = 0$$

$$g''(t) = f_{xx}(a+tu_1, b+tu_2)\, u_1 u_1$$

$$+ f_{yx}(a+tu_1, b+t \cdot u_2)\, u_1 u_2$$

$$+ f_{xy}(a+tu_1, b+tu_2)\, u_1 u_2$$

$$+ f_{yy}(a+tu_1, b+tu_2)\, u_2 u_2.$$

$$g''(0) = f_{xx}(a,b)\, u_1^2 +$$

$$\underbrace{f_{yx}}(a,b)\, u_1 u_2 + \underbrace{f_{xy}}(a,b)\, u_1 u_2$$

$$+ f_{yy}(a,b)\, u_2^2 .$$

$$g''(0) = f_{xx}(a,b)\, u_1^2 + 2 f_{yx}(a,b)\, u_1 u_2$$

$$+ f_{yy}(a,b)\, u_2^2 . > 0$$

for every (u_1, u_2)

$$\underline{\underline{a}}\,x^2 + 2b\,xy + c\,y^2 > 0$$

This happens if

$\boxed{a > 0}$ and

$$4b^2 - 4ac < 0$$

or $\boxed{ac - b^2 > 0}$

$$\begin{cases} \underline{f_{xx}(a,b) > 0} \\ D = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 > 0 \end{cases}$$

Whenever this happens we have a local minimum.

$$\begin{cases} f_{xx}(a,b) < 0 \\ D = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 > \end{cases}$$

$f(a,b)$ is a local maximum.

If
$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 < 0$$

We have a saddle point.

If $D = 0$ the test is inconclusive.

Example Find local maxima, minima and Saddle points of

$$f(x,y) = x^3 - 3x + 3xy^2.$$

Step 1: Find the critical points.

$$f_x(x,y)=0, \quad f_y(x,y)=0.$$

$$\begin{cases} f_x(x,y) = 3x^2 - 3 + 3y^2 = 0 \\ f_y(x,y) = 6xy = 0 \end{cases}$$

$xy = 0$ then

either $x = 0$ or $y = 0$.

<u>When $x = 0$</u>

$$3y^2 - 3 = 0$$
$$y^2 = 1$$
$$y = \pm 1$$

So we have two critical points

$(0, 1)$ and $(0, -1)$.

<u>When $y = 0$</u> : $3x^2 - 3 = 0$ $x^2 = 1$

$x = 1$ or $x = -1$

$(1, 0)$ and $(-1, 0)$.

Critical points:

$(0,1), (0,-1); (1,0)$

and $(-1,0)$.

<u>Step 2</u>: Which are max min or Saddle.

$f_x = 3x^2 - 3 + 3y^2$

$\boxed{f_{xx} = 6x}$, $\boxed{f_{xy} = 6y}$

$f_y = 6xy$

$\boxed{f_{yy} = 6x.}$

where

$D = f_{xx} f_{yy} - (f_{xy})^2$

$$\boxed{D = 36x^2 - 36y^2}$$

when $x = 0$, $y = \pm 1$

$D = -36 < 0$ saddle points

when $y = 0$, $x = \pm 1$

$$\boxed{D = 36 > 0.}$$

when $x = 1$ $f_{xx}(1,0) = 6 > 0.$

$f(1,0)$ min

$f_{xx}(-1,0) = -6 < 0$ max.